Quasi-Random Ideas. By Josef Dick.

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Walsh functions I. Orthonormality and completeness

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In this post I summarize some useful properties of Walsh functions. These functions were introduced by Joseph Walsh in


Another paper where many ideas can be found is by Nathan Fine


In this exposition here we only concentrate on the simplest case of base \( b = 2 \) and dimension \( s = 1 \).

We write \( \mathbb{N} \) for the set of natural numbers \( 1, 2, 3, \ldots \) and \( \mathbb{N}_0 \) for the set of nonnegative integers \( 0, 1, 2, \ldots \).

Definition of Walsh functions

**Definition (Walsh function)**

Let \( k \) be a nonnegative integer and \( x \in [0, 1) \) be a real number. Let the binary representation of \( k \) and \( x \) be given by

\[
\begin{align*}
k &= \kappa_0 + \kappa_1 2 + \cdots + \kappa_{d-1} 2^{a-1}, \\
x &= \frac{\alpha_0}{2} + \frac{\alpha_1}{2} + \cdots,
\end{align*}
\]

where \( \kappa_0, \kappa_1, \ldots, \kappa_{d-1}, \alpha_1, \alpha_2, \ldots \in \{0, 1\} \). Then the \( k \)th Walsh function

\[
\text{wal}_k : [0, 1) \to \{1, -1\}
\]
is given by
\[
\text{wal}_k(x) = (-1)^{\kappa_0 x_1 + \kappa_1 x_2 + \cdots + \kappa_{a-1} x_a}
\]
and further \( \text{wal}_0(x) = 1 \).

The following image shows the graph of \( \text{wal}_k \) for \( k = 1, 2, 3, 4 \):

First note that Walsh functions are piecewise constant. For \( k = 0 \) we have \( \text{wal}_0(x) = 1 \) for all \( x \in [0, 1) \).

Let \( k \in \mathbb{N} \) with binary representation \( k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{a-1} 2^{a-1} \). For \( x \in [r 2^{-a}, (r + 1) 2^{-a}) \) for some \( 0 \leq r < 2^a \) with binary representation \( r = r_0 + r_1 2 + \cdots + r_{a-1} 2^{a-1} \) we have
\[
x = \frac{r_0}{2} + \frac{r_1}{2^2} + \cdots + \frac{r_{a-1}}{2^a} + \frac{r_a}{2^{a+1}} + \cdots,
\]
\[
y = \frac{\kappa_0}{2} + \frac{\kappa_1}{2^2} + \cdots + \frac{\kappa_{a-1}}{2^a} + \frac{\kappa_a}{2^{a+1}} + \cdots.
\]

Then
\[
\text{wal}_k(x) = (-1)^{\kappa_0 y_0 + \kappa_1 y_1 + \cdots + \kappa_{a-1} y_{a-1}} = \text{wal}_k(y)
\]
and hence \( \text{wal}_k \) is constant on intervals of the form \([r 2^{-a}, (r + 1) 2^{-a})\).

**Walsh functions are orthonormal**

We now show that Walsh functions are orthogonal with respect to the \( L_2 \) inner product
\[
\langle f, g \rangle_{L_2} = \int_0^1 f(x)g(x) \, dx.
\]
where \( f, g : [0, 1] \to \mathbb{R} \) are real-valued functions. In the following we write \( \langle \cdot, \cdot \rangle \) instead of \( \langle \cdot, \cdot \rangle_{L_2} \).

Let \( k \in \mathbb{N}_0 \), then
\[
\langle \text{wal}_k, \text{wal}_k \rangle = \int_0^1 \text{wal}_k(x) \text{wal}_k(x) \, dx = \int_0^1 1 \, dx = 1.
\]

Now let \( k, l \in \mathbb{N}_0 \) with \( k \neq l \) have binary representations
\[
k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{a-1} 2^{a-1},
\]
\[
l = \lambda_0 + \lambda_1 2 + \cdots + \lambda_{b-1} 2^{b-1}.
\]
Without loss of generality assume that \( a \geq b \). Then
\[ \langle \text{wal}_k, \text{wal}_l \rangle = \int_0^1 \text{wal}_k(x) \text{wal}_l(x) \, dx \]
\[ = \frac{1}{2^k} \sum_{r=0}^{2^k-1} \text{wal}_k(2^{-r}x) \text{wal}_l(2^{-r}x) \]
\[ = \frac{1}{2^k} \sum_{r_0=0}^1 \cdots \sum_{r_{k-1}=0}^1 (-1)^{(\lambda_0 + \lambda_1 + \cdots + \lambda_{k-1})} \]
\[ = \prod_{i=0}^{k-1} \frac{1}{2} \sum_{r_{a-1}=0}^1 (-1)^{(\lambda_{a-1} + \lambda_a)} \]

Since \( k \neq l \), there is an \( 0 \leq i_0 < a \) such that \( \lambda_{i_0} + \lambda_{i_0} = 1 \) (where we set \( \lambda_0 = \cdots = \lambda_{a-1} = 0 \) if \( b < a \)) and therefore \( \sum_{r_{a-1}=0}^1 (-1)^{(\lambda_{a-1} + \lambda_a)} = 0 \). This implies that \( \langle \text{wal}_k, \text{wal}_l \rangle = 0 \)

**Proposition**
The Walsh functions \( \text{wal}_k \), where \( k = 0, 1, 2, \ldots \), are orthonormal with respect to the \( L_2 \) inner product, i.e. for all \( k, l \in \mathbb{N}_0 \) we have
\[ \langle \text{wal}_k, \text{wal}_l \rangle = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases} \]

**Walsh functions are complete**

We now show that the Walsh functions form a complete orthonormal system in \( L_2 \) (see the post on the mean square convergence of Fourier series for an analogous result for Fourier series).

**Definition (Walsh polynomial)**
Let \( N \in \mathbb{N}_0 \) and \( a_0, a_1, a_2, \ldots \in \mathbb{R} \) be real numbers. The function
\[ p : [0, 1] \to \mathbb{R} \]
\[ x \mapsto \sum_{k=0}^{N-1} a_k \text{wal}_k(x) \]

is called a Walsh polynomial.

As in the Fourier case, see here, we define the Dirichlet kernel for Walsh functions.

**Definition (Dirichlet kernel for Walsh functions)**
The Walsh polynomial
\[ D_N(x) = \sum_{k=0}^{N-1} \text{wal}_k(x) \]

is called the Walsh-Dirichlet kernel.

**Lemma**
For any \( m \in \mathbb{N}_0 \) we have
\[ D_{2^m}(x) = \begin{cases} 2^m & \text{if } x \in [0, 2^{-m}), \\ 0 & \text{if } x \in [2^{-m}, 1). \end{cases} \]

**Proof**
Let \( x = \frac{x_1}{2^1} + \frac{x_2}{2^2} + \cdots \). If \( x \in [0, 2^{-m}) \), then \( x_1 = \cdots = x_m = 0 \). Using the definition of the Dirichlet kernel we have
\[ D_{2m}(x) = \sum_{k=0}^{2^m-1} \text{wal}_k(x) \]
\[ = \sum_{k_0=0}^{1} \cdots \sum_{k_{m-1}=0}^{1} (-1)^{k_0 x_1 + \cdots + k_{m-1} x_m} = 2^m. \]

If \( x \in (2^{-m}, 1) \), then there is an \( 1 \leq i \leq m \) such that \( x_i = 1 \). Then
\[ D_{2m}(x) = \sum_{k=0}^{2^m-1} \text{wal}_k(x) \]
\[ = \sum_{k_0=0}^{1} \cdots \sum_{k_{m-1}=0}^{1} (-1)^{k_0 x_1 + \cdots + k_{m-1} x_m} \]
\[ = \sum_{k_0=0}^{1} \cdots \sum_{k_{m-1}=0}^{1} (-1)^{k_0 x_1 + \cdots + k_{m-1} x_m} = 0, \]
since \( \sum_{k_i=0}^{1} (-1)^{k_i x_i} = 0. \]

For \( x, y \in [0, 1) \) with \( x = \frac{u_1}{2} + \frac{u_2}{2^2} + \cdots \) and \( y = \frac{v_1}{2} + \frac{v_2}{2^2} + \cdots \) we define the digit-wise addition and subtraction by
\[ x \oplus y = \frac{u_1}{2} + \frac{u_2}{2^2} + \cdots \quad \text{and} \quad x \ominus y = \frac{v_1}{2} + \frac{v_2}{2^2} + \cdots, \]
where \( u_i, v_i \in \{0, 1\} \) are given by \( u_i = x_i + y_i \pmod 2 \) and \( v_i = x_i - y_i \pmod 2 \).

An immediate extension of the above lemma is that for any \( x, y \in [0, 1) \), where \( y \in [a2^{-m}, (a+1)2^{-m}] \) for some \( 0 \leq a < 2^{m} \), we have
\[ D_{2m}(x \oplus y) = \begin{cases} 2^m & \text{if } x \in [a2^{-m}, (a+1)2^{-m}), \\ 0 & \text{if } x \in [0, 1] \setminus [a2^{-m}, (a+1)2^{-m}). \end{cases} \]

The following result is equivalent to the completeness of the Walsh function system.

**Lemma**

Let \( f : [0, 1] \to \mathbb{R} \) be Lebesgue integrable such that
\[ \langle f, \text{wal}_k \rangle = \int_0^1 f(x) \text{wal}_k(x) \, dx = 0 \quad \text{for all } k \in \mathbb{N}_0. \]

Then \( f(x) = 0 \) for almost all \( x \in [0, 1] \).

**Proof**

Let \( F(x) = \int_0^x f(x) \, dx \) for \( x \in [0, 1] \).

Then by the fundamental theorem of calculus, \( F \) is continuous and differentiable almost everywhere with \( F' = f \). Then \( \langle f, \text{wal}_0 \rangle = 0 \) implies that \( F(1) = 0 \).

We proceed by induction on \( m \). Assume that for a given integer \( m \geq 0 \) we have shown already that \( F(a2^{-m}) = 0 \) for all \( 0 \leq a < 2^m \). For any \( 0 \leq b < 2^{m+1} \) we have
\[ \langle f, D_{2^{m+1}}(x \oplus b2^{-m-1}) \rangle = \int_{b2^{-m-1}}^{(b+1)2^{-m-1}} f(x) \, dx \]
\[ = F((b+1)2^{-m-1}) - F(b2^{-m-1}) = 0 \]

since \( \langle f, \text{wal}_k \rangle = 0 \) and therefore
\[ F(b2^{-m-1}) = F((b + 1)2^{-m-1}) \]

Since for \( b \) even we have \( F(b2^{-m-1}) = 0 \), it follows that \( F(b2^{-m-1}) = 0 \) for all \( 0 \leq b < 2^{m+1} \).

Hence \( F(a2^{-m}) = 0 \) for all \( 0 \leq a < 2^m \) and all \( m \in \mathbb{N} \). Since \( F \) is continuous and the numbers \( \{a2^{-m} : 0 \leq a < 2^m, m \in \mathbb{N} \} \) are dense in the interval \([0, 1]\) it follows that \( F \) is \( 0 \) everywhere. Further, \( \hat{f}(x) = F(x) \) almost everywhere, and hence \( f(x) = 0 \) for almost all \( x \in [0, 1] \).

**Definition**

Let \( f \in L_2([0, 1]) \). Then

\[ \hat{f}(k) = \langle f, \text{wal}_k \rangle \]

are called the **Walsh coefficients** of the function \( f \). The formal sum

\[ W(x) = \sum_{k=0}^{\infty} \hat{f}(k) \text{wal}_k(x) \]

is the **Walsh series** of \( f \). Further we define the partial sums

\[ W_K(x) = \sum_{k=0}^{K-1} \hat{f}(k) \text{wal}_k(x) \]

of the Walsh series of \( f \).

We define the \( L_2 \)-norm by \( \|f\|_{L_2} = \sqrt{\langle f, f \rangle_{L_2}} \).

**Theorem (Bessel's inequality)**

Let \( f \in L_2([0, 1]) \). Then

\[ \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \leq \|f\|_{L_2}^2. \]

**Proof**

Let \( 0 \leq k < K \), then

\[ \langle f - W_K, \text{wal}_k \rangle = \langle f, \text{wal}_k \rangle - \hat{f}(k) = 0 \]

Therefore \( f - W_K \) is orthogonal to \( W_K \) and we can use Pythagoras theorem to obtain

\[ \|f\|_{L_2}^2 = \|f - W_K + W_K\|_{L_2}^2 = \|f - W_K\|_{L_2}^2 + \|W_K\|_{L_2}^2 \geq \|W_K\|_{L_2}^2 \]

Since \( \|W_K\|_{L_2}^2 = \sum_{k=0}^{K-1} |\hat{f}(k)|^2 \) the result follows by considering the limit of \( K \to \infty \).

Note that for any \( f \in L_2([0, 1]) \), Bessel's inequality implies that \( W_1, W_2, \ldots \) is a Cauchy sequence ( \( \|W_M - W_K\|_{L_2} = \sum_{k=K}^{M-1} |\hat{f}(k)|^2 \) for \( M > K \)) and therefore converges to a function \( W \in L_2([0, 1]) \), with

\[ \|W\|_{L_2} = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \leq \|f\|_{L_2} < \infty. \]

**Theorem**

Let \( f \in L_2([0, 1]) \). Then

\[ \lim_{K \to \infty} \|f - W_K\|_{L_2} = 0 \]

**Proof**

Let \( g = f - W \). Then for all \( k \in \mathbb{N}_0 \) we have
\[ \langle g, \text{wal}_k \rangle = \langle f, \text{wal}_k \rangle - \langle W, \text{wal}_k \rangle = \hat{f}(k) - \hat{\phi}(k) = 0 \]

If \( \|g\|_{L^2} > 0 \), then set \( \phi(x) = g(x)/\|g\|_{L^2} \). Then \( \phi \) is orthogonal to all functions \( \text{wal}_k \), \( k \in \mathbb{N}_0 \) and \( \|\phi\|_{L^2} = 1 \). But this contradicts the lemma above and hence \( \lim_{K \to \infty} \|f - W_K\|_{L^2} = \|f - W\|_{L^2} = 0 \).

**Theorem (Parseval's identity)**

Let \( f \in L_2([0,1]) \). Then

\[ \sum_{k \in \mathbb{N}} |\hat{f}(k)|^2 = \|f\|^2_{L^2}. \]

**Proof**

We have

\[ \|f\|^2_{L^2} = \|f - W_K + W_K\|^2_{L^2} = \|f - W_K\|^2_{L^2} + \|W_K\|^2_{L^2}. \]

Since \( \|f - W_K\|_{L^2} \to 0 \) as \( K \to \infty \) the result follows.