Weighted Star Discrepancy of Digital Nets in Prime Bases

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Summary. We study the weighted star discrepancy of digital nets and sequences. Product weights and finite-order weights are considered and we prove tractability bounds for Niederreiter and Faure-Niederreiter sequences. Further we prove an existence result for digital nets achieving a strong tractability error bound by calculating the average over all generator matrices.

1 Introduction

For numerical integration of functions over the s-dimensional unit cube $[0,1)^s$ one needs point sets which are well distributed. What we mean by well distributed can be specified in several ways by various measures. Commonly such measures are based on the discrepancy function Δ which, for a point set $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ in the s-dimensional unit cube $[0,1)^s$, is defined by

$$\Delta(\alpha_1, \ldots, \alpha_s) := \frac{A_N\left(\prod_{i=1}^s [0, \alpha_i)\right)}{N} - \alpha_1 \cdots \alpha_s$$

for $0 < \alpha_1, \ldots, \alpha_s \leq 1$. Here $A_N(E)$ denotes the number of indices n, $0 \leq n \leq N-1$, such that $\boldsymbol{x}_n$ is contained in the set E. By taking the sup norm of this function, we obtain the *star discrepancy*

$$D_N^* = \sup_{\boldsymbol{z} \in (0,1]^s} |\Delta(\boldsymbol{z})|$$

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which appears in the Koksma-Hlawka inequality

$$\left| \int_{[0,1)^s} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{1}{N} \sum_{k=0}^{N-1} f(\boldsymbol{x}_k) \right| \le D_N^* V(f), \quad (1)$$

where $V(f)$ denotes the variation of f in the sense of Hardy and Krause.

It has been shown that there exist *low-discrepancy sequences*, i.e., sequences for which the first N points satisfy

$$D_N^* \le C_s \frac{(\log N)^s}{N}, \quad (2)$$

with a constant C_s only dependent on the dimension s. For values of N of practical interest, bounds of the form (2), and thus low-discrepancy sequences, seem to be useful only for dimensions up to about 15. In practice, though, low-discrepancy sequences have been used successfully in much higher dimensions.

In order to understand why low-discrepancy sequences can still work in higher dimensions, Sloan and Woźniakowski [15] (see also [5]) introduced a weighted discrepancy. The idea is that in many applications some projections are more important than others and that this should also be reflected in the quality measure of the point set. Before we give the definition of the weighted star discrepancy, let us introduce some notation.

Let $I_s = \{1, 2, \ldots, s\}$ denote the set of coordinate indices. For $u \subseteq I_s$, $u \neq \emptyset$, let $\gamma_{u,s}$ be a nonnegative real number (the weight), $|u|$ the cardinality of u, and for a vector $\boldsymbol{z} \in [0,1]^s$ let $\boldsymbol{z}_u$ denote the vector from $[0,1]^{|u|}$ containing the components of $\boldsymbol{z}$ whose indices are in u. By $(\boldsymbol{z}_u, 1)$ we mean the vector $\boldsymbol{z}$ from $[0,1]^s$ with all components whose indices are not in u replaced by 1.

Definition 1. For a point set $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ in $[0,1)^s$ and given weights $\gamma = \{\gamma_{u,s} : u \subseteq I_s, u \neq \emptyset\}$, the *weighted star discrepancy* $D_{N,\gamma}^*$ is given by

$$D_{N,\gamma}^* = \sup_{\boldsymbol{z} \in (0,1]^s} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \gamma_{u,s} |\Delta(\boldsymbol{z}_u, 1)|.$$

This is a generalization of the classical star discrepancy which is recovered if we choose $\gamma_{I_s,s} = 1$ and $\gamma_{u,s} = 0$ for all $\emptyset \neq u \subset I_s$. To avoid a trivial case, we will always assume that not all weights are 0. Furthermore the error bound (1) can also be generalized by replacing the star discrepancy with the weighted star discrepancy and the variation by a weighted version of the variation (see [15] for more details).

Now consider for example the case where $\gamma_{u,s} = 0$ for all $|u| \geq 3$ and $\gamma_{u,s} > 0$ for $|u| = 1, 2$. In this case it is reasonable to guess that there are point sets which achieve

$$D_{N,\gamma}^* \le C_{\gamma,s} \frac{(\log N)^2}{N},$$

where the constant depends only on the weights γ and the dimension s (such an example was also discussed in [5]). For such a choice of weights we can obtain useful error bounds also for a large dimension s (say several hundred), far beyond the previous suggestion that low-discrepancy sequences work only for dimensions up to about 15.

It is the aim of the paper to show that known point sets and sequences can indeed work well (meaning that we obtain a convergence rate of almost $O(N^{-1})$) in high dimensions under certain conditions on the weights. The results established here are of course for weights of a more general form than the special case considered above. Specifically, we will deal with two important kinds of weights, namely *product weights* and *finite-order weights*.

- Product weights are weights of the form $\gamma_{u,s} = \prod_{j\in u} \gamma_{j,s}$, for $u \subseteq I_s$, $u \neq \emptyset$, where $\gamma_{j,s}$ is the weight associated to the j-th component. Sometimes the weights $\gamma_{j,s}$ have no dependence on s, i.e., $\gamma_{j,s} = \gamma_j$. See [15, 5].
- Finite-order weights of order k, $k \in \mathbb{N}$ fixed, are weights with $\gamma_{u,s} = 0$ for all $u \subseteq I_s$ with $|u| > k$. See [6, 14].

Similar results for different cases have previously been established in papers such as [2, 14, 17, 18, 19], where either different point sets, discrepancies and settings have been considered. In those papers tractability and strong tractability has been investigated. Tractability can be shown by proving an upper bound on the discrepancy depending at most polynomially on the dimension, whereas strong tractability can be shown by proving an upper bound on the discrepancy which is independent of the dimension (for a formal definition of (strong) tractability see [15]).

We give a brief outline of the paper. In the following section we will introduce digital (t, m, s)-nets in base p. In Section 3 we introduce the tools needed to obtain bounds on the star discrepancy of digital nets. Section 4 deals with the weighted star discrepancy of digital nets and in Section 5 we obtain improved results for the special case where $p = 2$. Bounds on the weighted star discrepancy of Niederreiter and Faure-Niederreiter sequences are obtained in Section 6. The last section, Section 7, is concerned with existence results for digital nets satisfying a certain bound on the weighted star discrepancy.

2 Digital (t, m, s)-Nets in Base p

A detailed theory of (t, m, s)-nets was developed in Niederreiter [9] (see also [11, Chapter 4] and [12, Chapter 8] for surveys of this theory). We refer to [11] and [12] for the definition of (t, m, s)-nets. The crucial fact is that (t, m, s)-nets in a base p provide sets of p^m points in the s-dimensional unit cube $[0, 1)^s$ which are extremely well distributed if the quality parameter t is 'small'. From now on let p denote a prime number. We recall the following definition.

Definition 2. Let $p \geq 2$ be a given prime number and let $\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ be the finite field with p elements. Further let C_i, $i = 1, \ldots, s$, be s given

$m \times m$ matrices over $\mathbb{Z}_p$. Now we construct p^m points in $[0,1)^s$: represent $n \in \mathbb{Z}$, $0 \le n < p^m$, in base p, $n = n_0 + n_1 p + \cdots + n_{m-1}p^{m-1}$, and identify n with the vector $\mathbf{n} = (n_0, \ldots, n_{m-1})^T \in \mathbb{Z}_p^m$, where T means the transpose of the vector. For $1 \le i \le s$ multiply the matrix C_i by $\mathbf{n}$ modulo p,

$$C_i \mathbf{n} =: (y_{i,1}, \ldots, y_{i,m})^T \in \mathbb{Z}_p^m,$$

and set

$$x_{n,i} := \frac{y_{i,1}}{p} + \cdots + \frac{y_{i,m}}{p^m}.$$

If for some integer t with $0 \le t \le m$ the point set consisting of the points $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \le n < p^m$ is a (t,m,s)-net in base p, then it is called a *digital* (t,m,s)*-net in base* p (or *over* $\mathbb{Z}_p$), or shortly a *digital net* (*over* $\mathbb{Z}_p$).

3 The Star Discrepancy of Digital Nets

In this section we will introduce the tools needed for analyzing the star discrepancy of digital nets. The quantity $R_p(C_1, \ldots, C_s)$, defined in the following, will be useful to obtain bounds on the star discrepancy (compare with [11, Lemma 4.32]) as it captures the essential part of the discrepancy. In detail, we define

$$R_p(C_1, \ldots, C_s) := \sum_{\boldsymbol{k} \in \mathcal{D}} \prod_{i=1}^{s} r_p(k_i),$$

where the set $\mathcal{D}$ is the dual net restricted to $\{0, 1, \ldots, p^m - 1\}^s \setminus \{\mathbf{0}\}$, that is,

$$\mathcal{D} = \{\boldsymbol{k} \in \{0, 1, \ldots, p^m - 1\}^s \setminus \{\mathbf{0}\} : C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}\}, \tag{3}$$

$\boldsymbol{k} = (k_1, \ldots, k_s)$, $\mathbf{k}_i = (\kappa_0^{(i)}, \ldots, \kappa_{m-1}^{(i)})^T$ if $k_i = \sum_{j=0}^{m-1} \kappa_j^{(i)} p^j$, $1 \le i \le s$, and

$$r_p(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{1}{p^{g+1} \sin\left(\frac{\pi}{p}\kappa_g\right)} & \text{if } k = \kappa_0 + \kappa_1 p + \cdots + \kappa_g p^g,\ \kappa_g \neq 0. \end{cases}$$

Theorem 1. *For the star discrepancy D_N^* of the digital net over $\mathbb{Z}_p$ generated by the $m \times m$ matrices $C_1, \ldots, C_s$ we have, with $N = p^m$,*

$$D_N^* \le 1 - \left(1 - \frac{1}{N}\right)^s + R_p(C_1, \ldots, C_s) \le \frac{s}{N} + R_p(C_1, \ldots, C_s). \tag{4}$$

Proof. The result follows from the proof of [11, Lemma 4.32] by using [7, Theorem 1] instead of [11, Theorem 3.12]. □

We will obtain bounds on the star discrepancy by establishing bounds on R_p and using Theorem 1. It proves to be convenient to represent $R_p(C_1, \ldots, C_s)$ in terms of Walsh functions, which are introduced in the following.

For an integer $b \geq 2$ let $\omega_b = e^{2\pi \mathtt{i}/b} \in \mathbb{C}$. For a nonnegative integer k with base b representation $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$, the function ${}_b\mathrm{wal}_k : \mathbb{R} \longrightarrow \mathbb{C}$, periodic with period 1, is defined by

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\kappa_0 x_1 + \cdots + \kappa_r x_{r+1}}, $$

where $x \in [0,1)$ has base b representation $x = x_1/b + x_2/b^2 + \ldots$ (unique in the sense that infinitely many of the x_j must be different from $b-1$). Information on Walsh functions can be found in [1, 13, 16]. In the following we will always consider Walsh functions in base p, and hence we will often write wal_k instead of ${}_p\mathrm{wal}_k$.

Subsequently we will make use of the following result (see e.g. [3]): for the digital net $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{p^m-1}\}$ with $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$, generated by the $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_p$, we have

$$ \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} {}_p\mathrm{wal}_{k_i}(x_n^{(i)}) = \begin{cases} 1 \text{ if } C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}, \\ 0 \text{ otherwise.} \end{cases} \tag{5} $$

In the following lemma we show how R_p can be represented by Walsh functions.

Lemma 1. *Let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{p^m-1}$ with $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$ for $0 \leq n < p^m$ be a digital net over $\mathbb{Z}_p$ generated by $C_1, \ldots, C_s$. Then we have*

$$ R_p(C_1, \ldots, C_s) = -1 + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \left(1 + \sum_{k=1}^{p^m-1} r_p(k) \mathrm{wal}_k(x_n^{(i)}) \right). $$

Proof. We have

$$ \begin{aligned} & -1 + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \left(1 + \sum_{k=1}^{p^m-1} r_p(k) \mathrm{wal}_k(x_n^{(i)}) \right) \\ &= -1 + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \sum_{k=0}^{p^m-1} r_p(k) \mathrm{wal}_k(x_n^{(i)}) \\ &= -1 + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \sum_{k_1,\ldots,k_s=0}^{p^m-1} \prod_{i=1}^{s} r_p(k_i) \mathrm{wal}_{k_i}(x_n^{(i)}) \\ &= -1 + \sum_{k_1,\ldots,k_s=0}^{p^m-1} \prod_{i=1}^{s} r_p(k_i) \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \mathrm{wal}_{k_i}(x_n^{(i)}) \\ &= \sum_{\substack{k_1,\ldots,k_s=0 \\ (k_1,\ldots,k_s) \neq (0,\ldots,0)}}^{p^m-1} \prod_{i=1}^{s} r_p(k_i) \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \mathrm{wal}_{k_i}(x_n^{(i)}) \\ &= \sum_{\boldsymbol{k} \in \mathcal{D}} \prod_{i=1}^{s} r_p(k_i) = R_p(C_1, \ldots, C_s), \end{aligned} $$

where we used formula (5). □

4 Weighted Star Discrepancy of Digital Nets

It follows easily from Definition 1 that for the weighted star discrepancy $D^*_{N,\gamma}$ of a point set $\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}$ in $[0,1)^s$ we have

$$\begin{aligned} D^*_{N,\gamma} &= \sup_{\boldsymbol{z}\in(0,1]^s} \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s}|\Delta(\boldsymbol{z}_u,1)| \le \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} \sup_{\boldsymbol{z}_u\in(0,1]^{|u|}} |\Delta(\boldsymbol{z}_u,1)| \\ &= \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} D^*_N(u), \end{aligned}$$

where $D^*_N(u)$ denotes the star discrepancy of the $|u|$-dimensional projection of the point set $\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}$ to the coordinates given by u.

If we consider a digital net over $\mathbb{Z}_p$, generated by $C_1,\ldots,C_s$, then for $u\subseteq I_s$, $u\neq\emptyset$, from (4) we obtain

$$D^*_N(u) \le 1-\left(1-\frac{1}{N}\right)^{|u|} + R_p((C_i)_{i\in u})$$

and for $u=\{u_1,\ldots,u_d\}$, $R_p((C_i)_{i\in u})$ is given by

$$R_p((C_i)_{i\in u}) = \sum_{\substack{k_1,\ldots,k_d=0\\ (k_1,\ldots,k_d)\neq(0,\ldots,0)\\ C^T_{u_1}\mathbf{k}_1+\cdots+C^T_{u_d}\mathbf{k}_d=\mathbf{0}}}^{2^m-1} \prod_{i=1}^d r_p(k_i).$$

This leads to the following result.

Theorem 2. *For the weighted star discrepancy $D^*_{N,\gamma}$ of a digital net over $\mathbb{Z}_p$ generated by the $m\times m$ matrices $C_1,\ldots,C_s$ we have*

$$\begin{aligned} D^*_{N,\gamma} &\le \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s}\left(1-\left(1-\frac{1}{N}\right)^{|u|}\right) + \widetilde{R}_{p,\gamma}(C_1,\ldots,C_s) \\ &\le \frac{1}{N}\max_{\substack{u\subseteq I_s\\ u\neq\emptyset}}(|u|\gamma_{u,s}) + \widetilde{R}_{p,\gamma}(C_1,\ldots,C_s), \end{aligned}$$

where $N=p^m$ and

$$\widetilde{R}_{p,\gamma}(C_1,\ldots,C_s) := \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} R_p((C_i)_{i\in u}).$$

Hence we are concerned with the quality of projections of a digital net. Thus, it is useful to define a quality parameter t_u, $u\subseteq I_s$ with $u\neq\emptyset$, of the projection of a digital net to the coordinates given by the set u, that is, for all $u\subseteq I_s$ with $u\neq\emptyset$ the projection of the digital net P to the coordinates given by u is a $(t_u,m,|u|)$-net. The following definition and theorem prove to be useful.

Definition 3. For $1 \le i \le s$ let $\mathbf{c}_j^{(i)} \in \mathbb{Z}_p^m$, $1 \le j \le m$, be the row vectors of the matrix C_i. For $u \subseteq I_s$, $u \neq \emptyset$, let $\rho_u(C_1, \ldots, C_s)$ be the largest integer d such that any system $\{\mathbf{c}_j^{(i)} : 1 \le j \le d_i, i \in u\}$ with $0 \le d_i \le m$ for $i \in u$ and $\sum_{i \in u} d_i = d$ is linearly independent in $\mathbb{Z}_p^m$. (Here the empty system is viewed as linearly independent.)

Theorem 3. *Let p be a prime and let $C_1, \ldots, C_s \in \mathbb{Z}_p^{m \times m}$ be the generating matrices of a digital net P. Then P is a digital (t, m, s)-net over $\mathbb{Z}_p$ and the quality parameter t_u of the projection of the net to the coordinates given by u is $t_u = m - \rho_u(C_1, \ldots, C_s)$.*

Proof. The result follows from [11, Theorem 4.28]. □

Now we can give bounds on $\widetilde{R}_{p,\gamma}$ with the help of the quantities ρ_u.

Theorem 4. *For $s \ge 2$ and any prime p, we have*

$$\max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \frac{\gamma_{u,s}}{p^{\rho_u(C_1,\ldots,C_s)+1}} \le \widetilde{R}_{p,\gamma}(C_1, \ldots, C_s)$$
$$\le \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \gamma_{u,s} \left(1 - \frac{1}{p}\right) \left((m+1)^{|u|} - \binom{\rho_u(C_1, \ldots, C_s) + |u|}{|u|}\right) \frac{k(p)^{|u|}}{p^{\rho_u(C_1,\ldots,C_s)}},$$

where $k(2) = 1$ and $k(p) = \csc(\pi/p)$ if $p > 2$.

Proof. The result follows from the definition of $\widetilde{R}_{p,\gamma}(C_1, \ldots, C_s)$ together with the proof of [11, Theorem 4.34]. □

Corollary 1. *Let P be a digital (t, m, s)-net over $\mathbb{Z}_p$ and for $u \subseteq I_s$, $u \neq \emptyset$, let t_u denote the quality parameter of the projection of P to the coordinates given by u. Then we have*

$$D_{N,\gamma}^* \le \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} (|u| \gamma_{u,s})$$
$$+ \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \gamma_{u,s} \left(1 - \frac{1}{p}\right) k(p)^{|u|} \left((m+1)^{|u|} - \binom{m - t_u + |u|}{|u|}\right) p^{t_u},$$

where $N = p^m$ and $k(p)$ is defined in Theorem 4.

Proof. The corollary follows from Theorem 2, Theorem 3, and Theorem 4. □

5 An Alternative Bound in the Binary Case

For $p = 2$ we present an alternative bound on the weighted star discrepancy of digital nets over $\mathbb{Z}_p$ (see Corollary 2 below) which is often better than the corresponding bound in Corollary 1.

Lemma 2. *Let $m \geq 2$ and $s \geq 2$ and let $C_1, \ldots, C_s$ be the generating matrices of a digital (t, m, s)-net over $\mathbb{Z}_2$. Then we have*

$$\sum_{\substack{k_1,\ldots,k_s=1 \\ C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}}}^{2^m-1} \prod_{i=1}^{s} r_2(k_i) \leq 2^{t-m} m^s \left(\frac{1}{2^s} + \frac{m+1}{m^2} \right).$$

Proof. We have

$$\Sigma := \sum_{\substack{k_1,\ldots,k_s=1 \\ C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}}}^{2^m-1} \prod_{i=1}^{s} r_2(k_i) = \sum_{v_1,\ldots,v_s=0}^{m-1} \prod_{i=1}^{s} \frac{1}{2^{v_i+1}} \underbrace{\sum_{k_1=2^{v_1}}^{2^{v_1+1}-1} \cdots \sum_{k_s=2^{v_s}}^{2^{v_s+1}-1}}_{C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}} 1.$$

From the proof of [4, Lemma 7] we find that

$$\underbrace{\sum_{k_1=2^{v_1}}^{2^{v_1+1}-1} \cdots \sum_{k_s=2^{v_s}}^{2^{v_s+1}-1}}_{C_1^T \mathbf{k}_1 + \cdots + C_s^T \mathbf{k}_s = \mathbf{0}} 1 \leq \begin{cases} 0 & \text{if } v_1 + \cdots + v_s \leq m - t - s, \\ 1 & \text{if } m - t - s + 1 \leq v_1 + \cdots + v_s \\ & \qquad \leq m - t, \\ 2^{v_1 + \cdots + v_s - m + t} & \text{if } v_1 + \cdots + v_s > m - t. \end{cases}$$

Therefore we obtain

$$\Sigma \leq \frac{1}{2^s} \sum_{\substack{v_1,\ldots,v_s=0 \\ m-t-s+1 \leq v_1 + \cdots + v_s \leq m-t}}^{m-1} \frac{1}{2^{v_1 + \cdots + v_s}} + \frac{1}{2^s} \sum_{\substack{v_1,\ldots,v_s=0 \\ v_1 + \cdots + v_s > m-t}}^{m-1} \frac{2^t}{2^m} =: \Sigma_1 + \Sigma_2.$$

Trivially we have $\Sigma_2 \leq 2^{t-s-m} m^s$. Further we have

$$\begin{aligned} \Sigma_1 &\leq \frac{1}{2^s} \sum_{l=\max(0,m-t-s+1)}^{m-t} \binom{l+s-1}{s-1} \frac{1}{2^l} \\ &\leq \frac{1}{2^s} \binom{m-t+s-1}{s-1} \sum_{l=\max(0,m-t-s+1)}^{m-t} \frac{1}{2^l} \leq 2^{t-m} \binom{m-t+s-1}{s-1}. \end{aligned}$$

We obtain

$$\Sigma = \Sigma_1 + \Sigma_2 \leq 2^{t-m} \frac{m^s}{2^s} + 2^{t-m} \binom{m-t+s-1}{s-1}.$$

Therefore

$$\Sigma \leq 2^{t-m} m^s \left(\frac{1}{2^s} + \frac{1}{m^s} \binom{m+s-1}{s-1} \right)$$

and using $m \geq 2$ and $s \geq 2$,

$$\frac{1}{2^s} + \frac{1}{m^s}\binom{m+s-1}{s-1} = \frac{1}{2^s} + \frac{1}{m^s}\prod_{i=1}^{s-1}\frac{m+i}{i} = \frac{1}{2^s} + \frac{1}{m}\prod_{i=1}^{s-1}\left(\frac{1}{i}+\frac{1}{m}\right)$$
$$\leq \frac{1}{2^s} + \frac{1}{m}\left(1+\frac{1}{m}\right).$$

This yields the desired result. □

Remark 1. It is clear that for large m and s the bound in Lemma 2 can be improved by using sharper bounds for the product $\prod_{i=1}^{s-1}(i^{-1}+m^{-1})$ appearing in the last part of the proof.

Theorem 5. *Let $m \geq 2$, $N = 2^m$, $s \geq 2$ and $C_1, \ldots, C_s$ be the generating matrices of a digital (t, m, s)-net over $\mathbb{Z}_2$. For $u \subseteq I_s$, $u \neq \emptyset$, let t_u denote the quality parameter of the projection of the net to the coordinates given by u. Then we have*

$$\widetilde{R}_{2,\gamma}(C_1, \ldots, C_s)$$
$$\leq \frac{1}{N}\max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} 2^{t_u}\left(\left(\frac{m}{2}+1\right)^{|u|} - 1 + \frac{m+1}{m^2}\left((m+1)^{|u|}-1\right)\right).$$

Proof. We have

$$\widetilde{R}_{2,\gamma}(C_1, \ldots, C_s) = \max_{\substack{u\subseteq I_s\\ u\neq\emptyset\\ u=\{u_1,\ldots,u_d\}}} \gamma_{u,s} \sum_{\substack{k_1,\ldots,k_d=0\\ (k_1,\ldots,k_d)\neq(0,\ldots,0)\\ C_{u_1}^T\mathbf{k}_1+\cdots+C_{u_d}^T\mathbf{k}_d=\mathbf{0}}}^{2^m-1} \prod_{i=1}^{d} r_2(k_i)$$
$$= \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} \sum_{\substack{w\subseteq u\\ w\neq\emptyset\\ w=\{w_1,\ldots,w_e\}}} \sum_{\substack{k_1,\ldots,k_e=1\\ C_{w_1}^T\mathbf{k}_1+\cdots+C_{w_e}^T\mathbf{k}_e=\mathbf{0}}}^{2^m-1} \prod_{i=1}^{e} r_2(k_i).$$

Now we use Lemma 2 and obtain

$$\widetilde{R}_{2,\gamma}(C_1, \ldots, C_s) \leq \max_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_{u,s} \sum_{\substack{w\subseteq u\\ w\neq\emptyset}} 2^{t_w-m} m^{|w|}\left(\frac{1}{2^{|w|}} + \frac{m+1}{m^2}\right).$$

For $\emptyset \neq w \subseteq u$ we have $t_w \leq t_u$ and hence

$$\sum_{\substack{w\subseteq u\\ w\neq\emptyset}} 2^{t_w} m^{|w|}\left(\frac{1}{2^{|w|}} + \frac{m+1}{m^2}\right) \leq 2^{t_u}\sum_{d=1}^{|u|}\binom{|u|}{d}\left(\left(\frac{m}{2}\right)^d + \frac{m+1}{m^2}m^d\right)$$
$$= 2^{t_u}\left(\left(\frac{m}{2}+1\right)^{|u|} - 1 + \frac{m+1}{m^2}\left((m+1)^{|u|}-1\right)\right).$$

The result follows. □

Hence by using Theorems 2 and 5 we obtain the following improved upper bound on the weighted star discrepancy of a digital net over $\mathbb{Z}_2$.

Corollary 2. *Let $m \geq 2$, $N = 2^m$ and $s \geq 2$. Then for the weighted star discrepancy of a digital (t,m,s)-net over $\mathbb{Z}_2$ we have*

$$D^*_{N,\gamma} \leq \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \gamma_{u,s} 2^{t_u} \left(\left(\frac{m}{2} + 1 \right)^{|u|} - 1 + \frac{m+1}{m^2} ((m+1)^{|u|} - 1) \right).$$

6 Weighted Star Discrepancy of Niederreiter and Faure-Niederreiter Sequences

Weighted types of discrepancies of several types of sequences in a worst-case or randomized setting have previously been studied in papers such as [2, 14, 17, 18, 19]. Here we concentrate on the weighted star discrepancy of so-called Niederreiter and Faure-Niederreiter sequences using various types of weights as mentioned in the introduction. Most previously obtained results concentrate on the L_2 discrepancy, but a result directly comparable to a result in this paper (see Theorem 6 below) was obtained in [18, Lemma 3]. Further note that the results shown in this section can be used to obtain bounds on other discrepancies, as it was previously done in [14, 18].

In this section we apply the results from the previous section to give estimates for the weighted star discrepancy of Niederreiter sequences in a prime base p. The first p^m points of such sequences are a digital (t,m,s)-net over $\mathbb{Z}_p$, where the generating matrices are constructed in a special way and the quality parameter t is independent of m. For the general definition of Niederreiter sequences we refer to [10], [11, Section 4.5]. Here it suffices to note that the construction of these sequences depends on the choice of s distinct monic irreducible polynomials $q_1, \ldots, q_s$ over $\mathbb{Z}_p$. Then we get a (t,s)-sequence in base p with

$$t = \sum_{i=1}^{s} (\deg(q_i) - 1).$$

Let $u \subseteq I_s$, $u \neq \emptyset$, then we have that the projection of the sequence to the coordinates given by u is a $(t_u, |u|)$-sequence in base p with

$$t_u = \sum_{j \in u} (\deg(q_j) - 1). \tag{6}$$

As in [11] we consider two different choices for the polynomials $q_1, \ldots, q_s$. First we order the set of all monic irreducible polynomials over $\mathbb{Z}_p$ according to their degree such that $\deg(q_1) \leq \deg(q_2) \leq \ldots$. Let $\log_p$ denote the logarithm in base p. The following lemma was proved in [18, Lemma 2].

Lemma 3. *The degree of the j-th monic irreducible polynomial q_j over the finite field $\mathbb{Z}_p$ can be bounded by*

$$\deg(q_j) \le \log_p j + \log_p \log_p (j+p) + 2 \qquad \text{for } j = 1, 2, \ldots.$$

Thus, using (6), the value t_u can be bounded by

$$t_u = \sum_{j \in u} (\deg(q_j) - 1) \le \sum_{j \in u} (\log_p j + \log_p \log_p (j+p) + 1). \tag{7}$$

This yields the following discrepancy bound.

Theorem 6. *For the weighted star discrepancy of the first $N = p^m$ points of a Niederreiter (t,s)-sequence in prime base p we have*

$$D^*_{N,\gamma} \le \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} \gamma_{u,s} \prod_{j \in u} \frac{j p^2 \log_p (j+p) \log_p (pN)}{2}.$$

Proof. From Corollary 1 we obtain

$$D^*_{N,\gamma} \le \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} \gamma_{u,s} (m+1)^{|u|} k(p)^{|u|} p^{t_u}.$$

From $k(p) \le p/2$ and (7) we obtain

$$\begin{aligned} D^*_{N,\gamma} &\le \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} (|u| \gamma_{u,s}) \\ &\quad + \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} \gamma_{u,s} (m+1)^{|u|} \left(\frac{p}{2}\right)^{|u|} \prod_{j \in u} p^{\log_p j + \log_p \log_p (j+p) + 1} \\ &= \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{p^m} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} \gamma_{u,s} (m+1)^{|u|} \prod_{j \in u} \frac{j p^2 \log_p (j+p)}{2} \\ &= \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \ne \emptyset}} \gamma_{u,s} \prod_{j \in u} \frac{j p^2 \log_p (j+p) \log_p (pN)}{2}, \end{aligned}$$

which is the desired result. □

Remark 2. We remark that for $p = 2$ the bound in Theorem 6 can be improved if we use Corollary 2 instead of Corollary 1 in the proof.

Note that different types of bounds on the weighted star discrepancy of the Niederreiter sequence have previously been shown in [14, 18]. The bound in Theorem 6 can be used to show [18, Lemma 3]. In this case our bound improves the constant explicitly stated in the proof of [18, Lemma 3].

We obtain the following results for the product weight case and the finite-order weight case.

Corollary 3. *Assume that for $u \subseteq I_s$, $u \neq \emptyset$, $\gamma_{u,s}$ is given by $\gamma_{u,s} = \prod_{j \in u} \gamma_{j,s}$ with nonnegative reals $\gamma_{j,s}$. Then we have:*

1. *For the weighted star discrepancy of the first $N = p^m$ points of a Niederreiter (t,s)-sequence in prime base p we have*

$$D^*_{N,\gamma} \leq \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} (|u| \gamma_{u,s}) + \frac{1}{N} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \prod_{j \in u} \gamma_{j,s} \frac{j p^2 \log_p(j+p) \log_p(pN)}{2}.$$

2. *If*

$$\Gamma := \sup_{s \in \mathbb{N}} \max_{\substack{u \subseteq I_s \\ u \neq \emptyset}} \prod_{j \in u} \gamma_{j,s} \frac{j p^2 \log_p(j+p)}{2} < \infty, \tag{8}$$

then for the weighted star discrepancy of the first $N = p^m$ points of a Niederreiter (t,s)-sequence in prime base p we have

$$D^*_{N,\gamma} \leq 2\Gamma \frac{(m+1)^s}{N}.$$

Corollary 4. *Let $\{\gamma_{u,s}\}$ be arbitrary finite-order weights of order k. Then we have:*

1. *For the weighted star discrepancy of the first $N = p^m$ points of a Niederreiter (t,s)-sequence in prime base p we have*

$$D^*_{N,\gamma} \leq \frac{s^k}{N} c_{\gamma,s} \left(1 + \frac{p^2}{2} \log_p(s+p) \log_p(pN)\right)^k,$$

where $c_{\gamma,s} = \max_{\substack{u \subseteq I_s \\ 1 \leq |u| \leq k}} \gamma_{u,s}$.

2. *If*

$$\Gamma := \sup_{s \in \mathbb{N}} \max_{\substack{u \subseteq I_s \\ 1 \leq |u| \leq k}} \gamma_{u,s} \prod_{j \in u} j \log_p(j+p) < \infty, \tag{9}$$

then for any $\delta > 0$ there exists a $C_{p,\gamma,\delta} > 0$, independent of s and m, such that for the weighted star discrepancy of the first $N = p^m$ points of a Niederreiter (t,s)-sequence in prime base p we have

$$D^*_{N,\gamma} \leq \frac{C_{p,\gamma,\delta}}{N^{1-\delta}}.$$

Thus under condition (9) the upper bound is independent of the dimension which shows that the weighted star discrepancy of Niederreiter sequences achieves strong tractability.

Proof. From the proof of Theorem 6 and since we deal with finite-order weights of order k, we obtain

$$\begin{aligned} D^*_{N,\gamma} &\le \frac{1}{N} \max_{\substack{u\subseteq I_s\\ 1\le|u|\le k}} (|u|\gamma_{u,s}) + \frac{1}{N} \max_{\substack{u\subseteq I_s\\ 1\le|u|\le k}} \gamma_{u,s} \prod_{j\in u} \frac{jp^2 \log_p(j+p)\log_p(pN)}{2} \\ &\le \frac{1}{N} k c_{\gamma,s} + \frac{1}{N} \max_{\substack{u\subseteq I_s\\ 1\le|u|\le k}} \gamma_{u,s} \left(s\frac{p^2}{2} \log_p(s+p)\log_p(pN)\right)^k . \end{aligned}$$

The first part of the corollary follows. We prove the second part:

$$\begin{aligned} D^*_{N,\gamma} &\le \frac{1}{N} \max_{\substack{u\subseteq I_s\\ 1\le|u|\le k}} (|u|\gamma_{u,s}) + \frac{1}{N} \max_{\substack{u\subseteq I_s\\ 1\le|u|\le k}} \gamma_{u,s} \prod_{j\in u} \frac{jp^2 \log_p(j+p)\log_p(pN)}{2} \\ &\le \frac{\Gamma}{N} + \frac{1}{N} \max_{1\le r\le k} \left(\frac{p^2}{2}\log_p(pN)\right)^r \max_{\substack{u\subseteq I_s\\ |u|=r}} \gamma_{u,s} \prod_{j\in u} j \log_p(j+p) \\ &\le \frac{\Gamma}{N} + \frac{\Gamma}{N} \left(\frac{p^2}{2}\log_p(pN)\right)^k . \end{aligned}$$

The result follows. □

The first part of Corollary 4 shows that for finite order weights of order k we can indeed obtain a convergence rate of $O((\log N)^k N^{-1})$, as outlined in the example in the introduction. The second part, on the other hand, shows that under a certain condition on the weights we can even obtain an error bound independent of the dimension. The conditions on the weights are of a similar form as obtained in the earlier papers [14, 17, 18, 19], with the exception of [2] (see also Section 7 in this paper) where a much weaker dependence on the weights was shown. On the other hand, the construction in [2] requires a computer search and it is not known if a-priori given sequences can achieve such upper bounds on the star discrepancy.

If p is prime and s is an arbitrary dimension $\le p$, then for $q_1,\dots,q_s$ we can choose the polynomials $q_i(x) = x - a_i$, $1\le i\le s$, where $a_1,\dots,a_s$ are distinct elements of $\mathbb{Z}_p$; see [11, Remark 4.52]. (Sequences constructed with such polynomials are called Faure-Niederreiter sequences.) From (6) it follows that for any $u\subseteq I_s$, $u\ne\emptyset$, we have $t_u = 0$. This leads to the following result.

Theorem 7. *For the weighted star discrepancy of the first $N = p^m$ points of a Faure-Niederreiter $(0,s)$-sequence in prime base $p\ge s$ we have*

$$D^*_{N,\gamma} \le \frac{1}{N} \max_{\substack{u\subseteq I_s\\ u\ne\emptyset}} (|u|\gamma_{u,s}) + \frac{1}{N} \max_{\substack{u\subseteq I_s\\ u\ne\emptyset}} \gamma_{u,s} \left(\frac{p}{2}\log_p(pN)\right)^{|u|} .$$

Proof. The result follows from Corollary 1. □

Corollary 5. *Let $\{\gamma_{u,s}\}$ be arbitrary finite-order weights of order k. Then we have:*

1. *For the weighted star discrepancy of the first $N = p^m$ points of a Faure-Niederreiter $(0, s)$-sequence in prime base $p \geq s$ we have*

$$D^*_{N,\gamma} \leq \frac{1}{N} G_{\gamma,s} \left(k + \left(\frac{p}{2} \log_p(pN) \right)^k \right),$$

   where $G_{\gamma,s} := \max_{\substack{u \subseteq I_s \\ 1 \leq |u| \leq k}} \gamma_{u,s}$.

2. *If*

$$\sup_{s \in \mathbb{N}} \max_{\substack{u \subseteq I_s \\ 1 \leq |u| \leq k}} \gamma_{u,s} s^{|u|} < \infty, \tag{10}$$

   then for any $\delta > 0$ there exists a $C_{\gamma,\delta} > 0$, independent of s and m, such that for the first $N = p^m$ points of a Faure-Niederreiter $(0, s)$-sequence in prime base p, $s \leq p \leq 2s$, we have

$$D^*_{N,\gamma} \leq \frac{C_{\gamma,\delta}}{N^{1-\delta}}.$$

   Thus under condition (10) the upper bound is independent of the dimension which shows that the weighted star discrepancy of Faure-Niederreiter sequences achieves strong tractability.

3. *If*

$$\sup_{s \in \mathbb{N}} \max_{\substack{u \subseteq I_s \\ 1 \leq |u| \leq k}} \gamma_{u,s} < \infty, \tag{11}$$

   then for any $\delta > 0$ there exists a $C_{\gamma,\delta} > 0$, independent of s and m, such that for the first $N = p^m$ points of a Faure-Niederreiter $(0, s)$-sequence in prime base p, $s \leq p \leq 2s$, we have

$$D^*_{N,\gamma} \leq C_{\gamma,\delta} \frac{s^k}{N^{1-\delta}}.$$

   Thus under condition (11) the upper bound depends only polynomially on the dimension which shows that the weighted star discrepancy of Faure-Niederreiter sequences achieves tractability.

Proof. The first statement follows immediately from Theorem 7. For the second and third statement we use again Theorem 7 and note that we choose $s \leq p \leq 2s$, which is possible by Bertrand's postulate. □

The upper bound in Corollary 5 for the weighted star discrepancy of the Faure-Niederreiter sequence shows a stronger dependence on the dimension than the result for the Niederreiter sequence. This is due to the dependence of the base p on the dimension s, that is, we have to demand that $p \geq s$. Still, as part 3 of Corollary 5 shows, we can prove that the weighted star discrepancy depends at most polynomially on the dimension under certain conditions on the weights.

Corollary 6. *Assume that for $u \subseteq I_s$, $u \neq \emptyset$, $\gamma_{u,s}$ is given by $\gamma_{u,s} = \prod_{j\in u} \gamma_{j,s}$ with nonnegative reals $\gamma_{j,s}$. Then we have:*

1. *For the weighted star discrepancy of the first $N = p^m$ points of a Faure-Niederreiter $(0, s)$-sequence in prime base p, $s \le p \le 2s$, we have*

$$D^*_{N,\gamma} \le \frac{1}{N} \max_{\substack{u\subseteq I_s\\u\neq\emptyset}} \left(|u| \prod_{j\in u} \gamma_{j,s} \right) + \frac{1}{N} \max_{\substack{u\subseteq I_s\\u\neq\emptyset}} \prod_{j\in u} (\gamma_{j,s} s \log_p(pN)).$$

2. *If*

$$\Gamma := \sup_{s\in\mathbb{N}} \max_{\substack{u\subseteq I_s\\u\neq\emptyset}} \prod_{j\in u} \gamma_{j,s} s < \infty,$$

then for the first $N = p^m$ points of a Faure-Niederreiter $(0, s)$-sequence in prime base p, $s \le p \le 2s$, we have

$$D^*_{N,\gamma} \le 2\Gamma \frac{(m+1)^s}{N}.$$

7 Average Weighted Star Discrepancy

In this section we obtain an upper bound on the average of the weighted star discrepancy over all digital nets constructed over $\mathbb{Z}_p$ in s dimensions and with p^m points. We consider only weights of product form where the weights are independent of the dimension, that is, for $u \subseteq I_s$, $u \neq \emptyset$, the weights $\gamma_{u,s}$ are given by $\gamma_{u,s} = \gamma_u = \prod_{j\in u} \gamma_j$ with nonnegative reals γ_j independent of s. Define

$$\widehat{R}_{p,\gamma}(C_1, \ldots, C_s) := \sum_{\substack{u\subseteq I_s\\u\neq\emptyset}} \gamma_u R_p((C_i)_{i\in u}).$$

Then it follows from Theorem 2 that for the weighted star discrepancy $D^*_{N,\gamma}$ of a digital net over $\mathbb{Z}_p$ generated by the matrices $C_1, \ldots, C_s$ we have

$$D^*_{N,\gamma} \le \frac{1}{N} \max_{\substack{u\subseteq I_s\\u\neq\emptyset}} (|u|\gamma_u) + \widehat{R}_{p,\gamma}(C_1, \ldots, C_s), \tag{12}$$

where $N = p^m$.

Lemma 4. *We have*

$$\widehat{R}_{p,\gamma}(C_1, \ldots, C_s) = \sum_{\boldsymbol{k}\in\mathcal{D}} \prod_{i=1}^{s} \widehat{r}_p(k_i, \gamma_i),$$

where $\mathcal{D}$ is defined by (3) and $\widehat{r}_p(k, \gamma)$ is defined by

$$\widehat{r}_p(k, \gamma) := \begin{cases} 1 + \gamma & \text{if } k = 0, \\ \gamma r_p(k) & \text{if } k \neq 0. \end{cases}$$

Proof. Let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{p^m-1}$ be the digital net generated by $C_1, \ldots, C_s$ and write $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$ for $0 \le n < p^m$. From Lemma 1 it follows that for $u \subseteq I_s$, $u \neq \emptyset$, we have

$$R_p((C_i)_{i\in u}) = -1 + \frac{1}{p^m}\sum_{n=0}^{p^m-1}\prod_{i\in u}\left(1+\sum_{k=1}^{p^m-1} r_p(k)\mathrm{wal}_k(x_n^{(i)})\right).$$

Thus, we obtain

$$\begin{aligned}
&\sum_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \gamma_u R_p((C_i)_{i\in u})\\
&= -\sum_{\substack{u\subseteq I_s\\ u\neq\emptyset}}\gamma_u + \sum_{\substack{u\subseteq I_s\\ u\neq\emptyset}} \frac{1}{p^m}\sum_{n=0}^{p^m-1}\prod_{i\in u}\gamma_i\left(1+\sum_{k=1}^{p^m-1} r_p(k)\mathrm{wal}_k(x_n^{(i)})\right)\\
&= -\left(-1+\prod_{i=1}^{s}(1+\gamma_i)\right)\\
&\quad +\frac{1}{p^m}\sum_{n=0}^{p^m-1}\left(-1+\prod_{i=1}^{s}\left(1+\gamma_i+\gamma_i\sum_{k=1}^{p^m-1} r_p(k)\mathrm{wal}_k(x_n^{(i)})\right)\right)\\
&= -\prod_{i=1}^{s}(1+\gamma_i) + \frac{1}{p^m}\sum_{n=0}^{p^m-1}\prod_{i=1}^{s}\left(\sum_{k=0}^{p^m-1}\widehat{r}_p(k,\gamma_i)\mathrm{wal}_k(x_n^{(i)})\right)\\
&= -\prod_{i=1}^{s}(1+\gamma_i) + \sum_{k_1,\ldots,k_s=0}^{p^m-1}\prod_{i=1}^{s}\widehat{r}_p(k_i,\gamma_i)\frac{1}{p^m}\sum_{n=0}^{p^m-1}\prod_{i=1}^{s}\mathrm{wal}_{k_i}(x_n^{(i)})\\
&= \sum_{\substack{k_1,\ldots,k_s=0\\ (k_1,\ldots,k_s)\neq(0,\ldots,0)}}^{p^m-1}\prod_{i=1}^{s}\widehat{r}_p(k_i,\gamma_i)\frac{1}{p^m}\sum_{n=0}^{p^m-1}\prod_{i=1}^{s}\mathrm{wal}_{k_i}(x_n^{(i)})\\
&= \sum_{\boldsymbol{k}\in\mathcal{D}}\prod_{i=1}^{s}\widehat{r}_p(k_i,\gamma_i),
\end{aligned}$$

where we used formula (5). □

Let $\mathcal{C}_p := \{(C_1, \ldots, C_s) : C_i \in \mathbb{Z}_p^{m\times m} \text{ for } i = 1, \ldots, s\}$. Then we define

$$A_p(m,s) := \frac{1}{|\mathcal{C}_p|}\sum_{(C_1,\ldots,C_s)\in\mathcal{C}_p}\widehat{R}_{p,\gamma}(C_1,\ldots,C_s), \tag{13}$$

i.e., $A_p(m,s)$ is the average of $\widehat{R}_{p,\gamma}$ taken over all s-tuples of $m\times m$ matrices over $\mathbb{Z}_p$.

Theorem 8. *Let $A_p(m,s)$ be defined by (13) and $N = p^m$. Then for $p = 2$ we have*

$$A_2(m,s) = \frac{1}{2^m}\left(\prod_{i=1}^{s}\left(1+\gamma_i\left(\frac{m}{2}+1\right)\right) - \prod_{i=1}^{s}(1+\gamma_i)\right),$$

and for $p > 2$ we have

$$A_p(m,s) \le \frac{1}{p^m}\left(\prod_{i=1}^{s}\left(1+2\gamma_i m\left(\frac{1}{\pi}\log p+\frac{1}{5}\right)\right) - \prod_{i=1}^{s}(1+\gamma_i)\right).$$

Proof. We have

$$\begin{aligned} A_p(m,s) &= \frac{1}{p^{m^2 s}} \sum_{(C_1,\ldots,C_s)\in\mathcal{C}_p} \sum_{\boldsymbol{k}\in\mathcal{D}} \prod_{i=1}^{s} \widehat{r}_p(k_i,\gamma_i) \\ &= \frac{1}{p^{m^2 s}} \sum_{\substack{k_1,\ldots,k_s=0 \\ (k_1,\ldots,k_s)\neq(0,\ldots,0)}}^{p^m-1} \prod_{i=1}^{s} \widehat{r}_p(k_i,\gamma_i) \sum_{\substack{(C_1,\ldots,C_s)\in\mathcal{C}_p \\ C_1^T \mathbf{k}_1+\cdots+C_s^T\mathbf{k}_s=\mathbf{0}}} 1. \end{aligned}$$

Let $\mathbf{c}_j^{(i)}$ denote the j-th row vector, $1 \le j \le m$, of the matrix C_i, $1 \le i \le s$. Then for $\boldsymbol{k} \in \{0,1,\ldots,p^m-1\}^s$, $\boldsymbol{k} \neq \mathbf{0}$, our condition in the innermost sum of the above expression becomes

$$\sum_{i=1}^{s}\sum_{j=0}^{m-1} \mathbf{c}_{j+1}^{(i)} \kappa_{i,j} = \mathbf{0}, \tag{14}$$

where $k_i = \kappa_{i,0}+\kappa_{i,1}p+\cdots+\kappa_{i,m-1}p^{m-1}$. Since at least one $k_i \neq 0$, it follows that there is a $\kappa_{i,j} \neq 0$. First assume that $\kappa_{1,0} \neq 0$. Then for any choice of

$$\mathbf{c}_2^{(1)},\ldots,\mathbf{c}_m^{(1)},\mathbf{c}_1^{(2)},\ldots,\mathbf{c}_m^{(2)},\ldots,\mathbf{c}_1^{(s)},\ldots,\mathbf{c}_m^{(s)}$$

we can find exactly one vector $\mathbf{c}_1^{(1)}$ such that condition (14) is fulfilled. The same argument holds with $\kappa_{1,0}$ replaced by $\kappa_{i,j}$ and $\mathbf{c}_1^{(1)}$ replaced by $\mathbf{c}_{j+1}^{(i)}$.

Therefore we get

$$\begin{aligned} A_p(m,s) &= \frac{1}{p^m} \sum_{\substack{k_1,\ldots,k_s=0 \\ (k_1,\ldots,k_s)\neq(0,\ldots,0)}}^{p^m-1} \prod_{i=1}^{s} \widehat{r}_p(k_i,\gamma_i) \\ &= \frac{1}{p^m}\left(\prod_{i=1}^{s}\sum_{k=0}^{p^m-1} \widehat{r}_p(k,\gamma_i) - \prod_{i=1}^{s}(1+\gamma_i)\right) \\ &= \frac{1}{p^m}\left(\prod_{i=1}^{s}\left(1+\gamma_i+\gamma_i\sum_{k=1}^{p^m-1} r_p(k)\right) - \prod_{i=1}^{s}(1+\gamma_i)\right) \\ &= \frac{1}{p^m}\left(\prod_{i=1}^{s}\left(1+\gamma_i\sum_{k=0}^{p^m-1} r_p(k)\right) - \prod_{i=1}^{s}(1+\gamma_i)\right). \end{aligned}$$

From the proof of [11, Lemma 3.13] we find that

$$\sum_{k=0}^{2^m-1} r_2(k) = \frac{m}{2} + 1$$

and that for $p > 2$,

$$\sum_{k=0}^{p^m-1} r_p(k) \leq m\left(\frac{2}{\pi}\log p + \frac{2}{5}\right).$$

The result follows. □

Corollary 7. *For any prime p and $\varepsilon \geq 1$ we have*

$$\left|\left\{(C_1,\ldots,C_s) \in \mathcal{C}_p : \widehat{R}_{p,\gamma}(C_1,\ldots,C_s) \leq \frac{\varepsilon}{N} H(p,m,s,\gamma)\right\}\right| \geq |\mathcal{C}_p|\left(1 - \frac{1}{\varepsilon}\right),$$

where $N = p^m$,

$$H(p,m,s,\gamma) := \prod_{i=1}^{s}(1+\gamma_i h(p,m)) - \prod_{i=1}^{s}(1+\gamma_i)$$

and where $h(p,m) = 2m((\log p)/\pi + 1/5)$ if $p \neq 2$ and $h(2,m) = m/2+1$.

Proof. From Theorem 8 we obtain

$$\begin{aligned}
&\frac{1}{p^m} H(p,m,s,\gamma) \geq \frac{1}{|\mathcal{C}_p|} \sum_{(C_1,\ldots,C_s)\in\mathcal{C}_p} \widehat{R}_{p,\gamma}(C_1,\ldots,C_s) \\
&\geq \frac{1}{|\mathcal{C}_p|}\frac{\varepsilon}{p^m} H(p,m,s,\gamma) \times \\
&\quad \left|\left\{(C_1,\ldots,C_s) \in \mathcal{C}_p : \widehat{R}_{p,\gamma}(C_1,\ldots,C_s) > \frac{\varepsilon}{p^m} H(p,m,s,\gamma)\right\}\right|.
\end{aligned}$$

Hence we have

$$\frac{1}{\varepsilon} \geq \frac{1}{|\mathcal{C}_p|}\left|\left\{(C_1,\ldots,C_s) \in \mathcal{C}_p : \widehat{R}_{p,\gamma}(C_1,\ldots,C_s) > \frac{\varepsilon}{p^m} H(p,m,s,\gamma)\right\}\right|,$$

and the result follows. □

Let $c_p := 2\left(\frac{1}{\pi}\log p + \frac{1}{5}\right)$, $N = p^m$, and assume that $\sum_{i=1}^{\infty}\gamma_i < \infty$. Then we have

$$\prod_{i=1}^{s}(1+\gamma_i m c_p) = \prod_{i=1}^{s}\left(1+\gamma_i(\log N)\frac{c_p}{\log p}\right) \leq \prod_{i=1}^{s}(1+\gamma_i c\log N),$$

where $c > 1$ is an absolute constant. Now we follow the proof of [8, Lemma 3]. Let

$$S(\gamma, N) := \prod_{i=1}^{\infty}(1 + \gamma_i c \log N)$$

and define $\sigma_d := c\sum_{i=d+1}^{\infty}\gamma_i$ for $d \geq 0$. Then

$$\begin{aligned}
\log S(\gamma, N) &= \sum_{i=1}^{\infty}\log(1 + \gamma_i c \log N) \\
&\leq \sum_{i=1}^{d}\log(1 + \sigma_d^{-1} + \gamma_i c \log N) + \sum_{i=d+1}^{\infty}\log(1 + \gamma_i c \log N) \\
&\leq d\log(1 + \sigma_d^{-1}) + \sum_{i=1}^{d}\log(1 + \gamma_i \sigma_d c \log N) \\
&\quad + \sum_{i=d+1}^{\infty}\log(1 + \gamma_i c \log N) \\
&\leq d\log(1 + \sigma_d^{-1}) + \sigma_d c(\log N)\sum_{i=1}^{d}\gamma_i + \sigma_d \log N \\
&\leq d\log(1 + \sigma_d^{-1}) + \sigma_d(\sigma_0 + 1)\log N.
\end{aligned}$$

Hence we obtain

$$S(\gamma, N) \leq (1 + \sigma_d^{-1})^d p^{m(\sigma_0+1)\sigma_d}.$$

For $\delta > 0$ choose d large enough to make $\sigma_d \leq \delta/(\sigma_0 + 1)$. Then we obtain

$$S(\gamma, N) \leq c_{\gamma,\delta}p^{\delta m}.$$

Therefore from (12) and Theorem 8 we obtain the following result.

Corollary 8. *If $\sum_{i=1}^{\infty}\gamma_i < \infty$, then for any $\delta > 0$ there exist a constant $\widetilde{c}_{\gamma,\delta} > 0$, independent of s and m, and $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_p$ such that the weighted star discrepancy $D^*_{N,\gamma}$ of the digital net generated by $C_1, \ldots, C_s$ satisfies*

$$D^*_{N,\gamma} \leq \frac{\widetilde{c}_{\gamma,\delta}}{N^{1-\delta}} \tag{15}$$

for all $m, s \geq 1$ and where $N = p^m$. Hence there exist digital nets whose weighted star discrepancy achieves a strong tractability error bound as long as the sum of the weights is finite.

In [2] the authors introduced an algorithm which shows how matrices $C_1, \ldots, C_s$ which satisfy a bound of the form (15) can be found by computer search.

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