Chapter 1

Constructing good lattice rules with millions of points

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Abstract W

e develop an algorithm for the construction of randomly shifted rank-1 lattice rules in weighted Sobolev spaces with a significantly reduced construction cost. The results shown here are an extension of earlier results by the present authors. In this new algorithm, the number of quadrature points n is a product of r distinct prime numbers $p_1, \ldots, p_r$. This allows us to reduce the construction cost to $O(n(p_1 + \cdots + p_r)d^2)$, which represents a significant reduction, especially for large n. The constructed rules achieve a worst-case error bound with a rate of convergence of $O(p_1^{-1+\delta} p_2^{-1/2} \cdots p_r^{-1/2})$ for any $\delta > 0$. Numerical experiments were carried out for $r = 2, 3, 4$ and 5. The results demonstrate that it can be advantageous to choose n as a product of up to 5 primes.

1.1 Introduction

In this paper we are interested in approximating the d-dimensional integral

$$I_d(f) = \int_{[0,1]^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

of functions f belonging to weighted Sobolev spaces. The tools we are using here are "randomly shifted rank-1 lattice rules" which are of the form:

$$Q_{n,d}(f, \boldsymbol{v}, \boldsymbol{\Delta}) = \frac{1}{n} \sum_{\ell=1}^{n} f\left(\left\{\frac{\ell \boldsymbol{v}}{n} + \boldsymbol{\Delta}\right\}\right),$$

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where $\boldsymbol{v} = (v_1, \ldots, v_d)$ is a d-dimensional integer vector called the "generating vector" and the "shift" $\boldsymbol{\Delta}$ is drawn randomly from a uniform distribution on $[0,1]^d$. Here the braces around a vector indicate that we take the fractional part of each component of the vector.

The weighted Sobolev spaces mentioned above are tensor products of 1-dimensional reproducing kernel Hilbert spaces of functions with square-integrable mixed first derivatives. These spaces are parameterized by a non-increasing sequence $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ of "weights", which moderate the behaviour of successive dimensions in those spaces (see [11] for more information about weighted spaces). Analysis of integration in such spaces was conducted in many papers (see for example [5, 6, 11, 12]).

In our subsequent analysis of randomly shifted rank-1 lattice rules in those weighted Sobolev spaces, we will study the "worst-case error" defined by (see [4, 5] for more information)

$$e_{n,d}^2(\boldsymbol{v}) := \int_{[0,1]^d} e_{n,d}^2(\boldsymbol{v}, \boldsymbol{\Delta}) \, \mathrm{d}\boldsymbol{\Delta},$$

where

$$e_{n,d}(\boldsymbol{v}, \boldsymbol{\Delta}) := \sup\{|I_d(f) - Q_{n,d}(f, \boldsymbol{v}, \boldsymbol{\Delta})| : f \in H_d, ||f||_d \leq 1\},$$

with $||\cdot||_d$ denoting the norm in H_d. This worst-case error is given explicitly by (see [9])

$$e_{n,d}^2(\boldsymbol{v}) = -\prod_{j=1}^{d}\left(1 + \frac{\gamma_j}{3}\right) + \frac{1}{n}\sum_{\ell=1}^{n}\prod_{j=1}^{d}\left(1 + \gamma_j\left[B_2\left(\left\{\frac{\ell \boldsymbol{v}}{n}\right\}\right) + \frac{1}{3}\right]\right), \quad (1.1)$$

where $B_2(x) = x^2 - x + 1/6$ is the Bernoulli polynomial of degree 2.

For any given generating vector $\boldsymbol{v}$, formula (1.1) allows us to calculate the worst-case error exactly with a computational cost of $O(nd)$ operations, and this in turn allows us to search for good generating vectors. However, even for moderate dimensions d and number of points n, a search over all possible generating vectors is too costly (the order of the search cost is n^d). In [10] Sloan and Reztsov introduced a construction method for generating vectors in unweighted Korobov spaces, making the construction feasible for large d and n. This method was extended in [9] to randomly shifted lattice rules in weighted Sobolev spaces. In these algorithms, the first component of the generating vector $\boldsymbol{v}$ is set to 1 and the remaining components are found step-by-step such that the worst-case error is minimized in each step. The construction cost of this method is $O(n^2d^2)$ operations.

Recently Dick and Kuo [2] introduced a modified algorithm. They considered the case where n is a product of two distinct primes p_1 and p_2. Choosing n in such a way allowed them to reduce the construction cost to $O(n(p_1 + p_2)d^2)$ operations. For $p_1 \approx p_2$ this reduction is by a factor of $n^{1/2}$, which for example yields a thousand-fold reduction for $n = 10^6$. Such a reduction in cost makes it possible to construct rules with millions of points, as shown in the numerical experiments therein.

Here we generalize the algorithm of [2] to the case where n is a product of r distinct prime numbers $p_1, \ldots, p_r$. (For $r = 1$ we obtain the algorithm of [9].) This is done in Section 1.2. The construction cost is now $O(n(p_1 + \cdots + p_r)d^2)$ operations, with a minimum of $O(n^{1+1/r}d^2)$ operations if we choose the primes to be roughly equal. This speedup in the construction of good lattice rule is significant. For example, when n is 10^6, the construction cost using three primes is reduced by a factor of 10 compared to the construction cost using two primes (and by a factor of 10000 compared to using just one prime). The construction cost can be reduced even further by using more primes.

Section 1.3 contains a bound on the worst-case error for rules constructed by our new algorithm. This bound is a generalization of a result in [2] (and in [1]) and is also obtained by an averaging argument. The theoretical rate of convergence for our rules is $O(p_1^{-1+\delta} p_2^{-1/2} \cdots p_r^{-1/2})$ for $\delta > 0$. When the primes are roughly equal, in which case the construction cost is minimized, the rate of convergence is $O(n^{-(1+1/r)/2+\delta})$ for $\delta > 0$. Clearly, this convergence rate gets worse as r gets larger.

The final section, Section 1.4, contains numerical experiments with n taken to be a product of 2, 3, 4 and 5 prime numbers and a maximum value n of about eight million. From the graphs presented there it is apparent that the observed rate of convergence does not deteriorate as we increase r. From those considerations it seems advisable to choose n as a product of up to 5 primes in appropriate cases.

1.2 The component-by-component algorithm

In [2] we considered rank-1 lattice rules with $n = p_1 p_2$ points, where p_1 and p_2 are distinct prime numbers. The quadrature points are given by the set

$$\left\{ \left\{ \frac{\ell_1 \boldsymbol{z}_1}{p_1} + \frac{\ell_2 \boldsymbol{z}_2}{p_2} \right\} : 1 \le \ell_1 \le p_1, 1 \le \ell_2 \le p_2 \right\},$$

where $\boldsymbol{z}_1 \in \{1, \ldots, p_1 - 1\}^d$ and $\boldsymbol{z}_2 \in \{1, \ldots, p_2 - 1\}^d$. This idea originates from [8] (see also [7]). Here we have a generalized algorithm in which n is a product of $r \ge 1$ distinct prime numbers $p_1, \ldots, p_r$, and we construct vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ with $\boldsymbol{z}_m \in \{1, \ldots, p_m - 1\}^d$ for each $m = 1, \ldots, r$. The generating vector $\boldsymbol{v} \in \{1, \ldots, n-1\}^d$ of our rank-1 lattice rule is then given by

$$\boldsymbol{v} \equiv \boldsymbol{z}_1 \frac{n}{p_1} + \cdots + \boldsymbol{z}_r \frac{n}{p_r} \pmod{n}. \tag{1.2}$$

We see from (1.2) that $\boldsymbol{v} \equiv \boldsymbol{z}_m n/p_m \not\equiv 0 \pmod{p_m}$ for each $m = 1, \ldots, r$, that is, the components of $\boldsymbol{v}$ have no factor in common with n. Moreover, it follows from the Chinese reminder theorem that there is a one-to-one correspondence between $\boldsymbol{v}$ and $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r)$ given by (1.2). For ℓ satisfying $1 \le \ell \le n$ we have

$$\left\{ \frac{\ell \boldsymbol{v}}{n} \right\} = \left\{ \frac{\ell_1 \boldsymbol{z}_1}{p_1} + \cdots + \frac{\ell_r \boldsymbol{z}_r}{p_r} \right\},$$

where $\ell_m \equiv \ell \pmod{p_m}$ for each $m = 1, \ldots, r$, and again by the Chinese remainder theorem, there is a one-to-one correspondence between such ℓ and $(\ell_1, \ldots, \ell_r)$. Based on these arguments we conclude that the set of points

$$\left\{ \left\{ \frac{\ell \boldsymbol{v}}{n} \right\} : 1 \le \ell \le n \right\},$$

is identical to the set

$$\left\{ \left\{ \frac{\ell_1 \boldsymbol{z}_1}{p_1} + \cdots + \frac{\ell_r \boldsymbol{z}_r}{p_r} \right\} : 1 \le \ell_m \le p_m, \text{ for each } m = 1, \ldots, r \right\}.$$

Thus by using (1.2) the worst-case error (1.1) can be written as

$$\begin{aligned} &e_{n,d}^2(\boldsymbol{v}) =: e_{n,d}^2(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r) \qquad (1.3) \\ &= -\prod_{j=1}^{d} \left(1 + \frac{\gamma_j}{3}\right) + \frac{1}{n} \sum_{\ell_1=1}^{p_1} \cdots \sum_{\ell_r=1}^{p_r} \prod_{j=1}^{d} \left(1 + \gamma_j \left[B_2\left(\left\{ \sum_{i=1}^{r} \frac{\ell_i z_{i,j}}{p_i} \right\} \right) + \frac{1}{3} \right] \right). \end{aligned}$$

Algorithm 1 given below generalizes the Partial Search algorithm in [2]. (For $r = 1$ we obtain the algorithm in [9].) In this new algorithm we first construct the vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ component-by-component. At each step, $z_{m,s}$ (the s-th component of the m-th vector $\boldsymbol{z}_m$) is found by minimizing some function $\Theta_{n,s}^{(m)}$ for each $m = 1, \ldots, r$. The s-th component v_s, satisfying $1 \le v_s \le n-1$, is then obtained from the equation $v_s \equiv z_{1,s} n/p_1 + \cdots + z_{r,s} n/p_r \pmod{n}$. For each $m = 1, \ldots, r-1$, $\Theta_{n,s}^{(m)}$ is defined to be the mean of $e_{n,s}^2$ over the components $z_{m+1,s}, \ldots, z_{r,s}$, that is,

$$\begin{aligned} &\Theta_{n,s}^{(m)}(\boldsymbol{z}_1^{(s-1)}, \ldots, \boldsymbol{z}_r^{(s-1)}; z_{1,s}, \ldots, z_{m,s}) \\ &:= \frac{1}{(p_{m+1}-1) \cdots (p_r - 1)} \sum_{z_{m+1,s}=1}^{p_{m+1}-1} \cdots \sum_{z_{r,s}=1}^{p_r - 1} e_{n,s}^2(\boldsymbol{z}_1^{(s)}, \ldots, \boldsymbol{z}_r^{(s)}), \qquad (1.4) \end{aligned}$$

and for $m = r$ we define

$$\Theta_{n,s}^{(r)}(\boldsymbol{z}_1^{(s-1)}, \ldots, \boldsymbol{z}_r^{(s-1)}; z_{1,s}, \ldots, z_{r,s}) := e_{n,s}^2(\boldsymbol{z}_1^{(s)}, \ldots, \boldsymbol{z}_r^{(s)}). \qquad (1.5)$$

For any d-dimensional vector $\boldsymbol{x} = (x_1, \ldots, x_d)$, the notation $\boldsymbol{x}^{(s)}$ for $s \le d$ is used to denote the s-dimensional vector $(x_1, \ldots, x_s)$.

Algorithm 1 [Partial Search]
Given distinct prime numbers $p_1, \ldots, p_r$, where $r \ge 1$:

1. *Set $\hat{z}_{1,1}, \ldots, \hat{z}_{r,1}$, the first components of $\hat{\boldsymbol{z}}_1, \ldots, \hat{\boldsymbol{z}}_r$, to 1.*
2. *For each $s = 2, \ldots, d$, do the following:*

   For each $m = 1, \ldots, r$, find a $\hat{z}_{m,s} \in \{1, \ldots, p_m - 1\}$ such that

$$\Theta_{n,s}^{(m)}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}; \hat{z}_{1,s}, \ldots, \hat{z}_{m-1,s}, z_{m,s}),$$

   as a function of $z_{m,s}$, is minimized.

3. *The generating vector $\hat{\boldsymbol{v}} \in \{1, \ldots, n-1\}^d$ is then given by*

$$\hat{\boldsymbol{v}} \equiv \hat{\boldsymbol{z}}_1 \frac{n}{p_1} + \ldots + \hat{\boldsymbol{z}}_r \frac{n}{p_r} \pmod{n}.$$

To be able to use Algorithm 1 we need an explicit expression for $\Theta_{n,s}^{(m)}$, which is given in the next theorem. We see from this theorem that it requires $O(ns)$ operations to compute $\Theta_{n,s}^{(m)}$. Therefore the construction cost for a d-dimensional generating vector $\hat{\boldsymbol{v}}$ using Algorithm 1 is $O(n(p_1 + \cdots + p_r)d^2)$ operations. Throughout the paper we will use the convention that the empty product is 1.

Theorem 1. *Let $r \ge 2$ and $n = p_1 \cdots p_r$ where $p_1, \ldots, p_r$ are distinct prime numbers. Then for each $m = 1, \ldots r-1$, we have*

$$\begin{aligned}
&\Theta_{n,s}^{(m)}(\boldsymbol{z}_1^{(s-1)}, \ldots, \boldsymbol{z}_r^{(s-1)}; z_{1,s}, \ldots, z_{m,s}) \\
&= \left(1 + \tfrac{\gamma_s}{3}\right) e_{n,s-1}^2(\boldsymbol{z}_1^{(s-1)}, \ldots, \boldsymbol{z}_r^{(s-1)}) \\
&\quad + \frac{\gamma_s}{n} \sum_{\substack{\mathcal{W} \subseteq \{1,\ldots,r\} \\ \mathcal{W} = \{i_1,\ldots,i_{|\mathcal{W}|}\}}} \sum_{\ell_{i_1}=1}^{p_{i_1}-1} \cdots \sum_{\ell_{i_{|\mathcal{W}|}}=1}^{p_{i_{|\mathcal{W}|}}-1} \left[\prod_{j=1}^{s-1} \left(1 + \gamma_j \left[B_2\left(\left\{ \sum_{i \in \mathcal{W}} \frac{\ell_i z_{i,j}}{p_i} \right\} \right) + \frac{1}{3} \right] \right) \right. \\
&\quad \times \prod_{i \in \mathcal{W} \setminus \{1,\ldots,m\}} (p_i - 1)^{-1} \\
&\quad \left. \times \sum_{\mathfrak{u} \subseteq \mathcal{W} \setminus \{1,\ldots,m\}} \left[\frac{(-1)^{|\mathfrak{u}| + |\mathcal{W} \setminus \{1,\ldots,m\}|}}{\prod_{i \in \mathfrak{u}} p_i} B_2\left(\left\{ \prod_{i \in \mathfrak{u}} p_i \sum_{i \in \mathcal{W} \cap \{1,\ldots,m\}} \frac{\ell_i z_{i,s}}{p_i} \right\} \right) \right] \right].
\end{aligned}$$

Proof. By separating out the cases $\ell_i = p_i$ for each $i = 1, \ldots, r$, we can write (1.3) as

$$
\begin{aligned}
&e_{n,s}^2(\boldsymbol{z}_1^{(s)},\ldots,\boldsymbol{z}_r^{(s)})\\
&= -\prod_{j=1}^{s}\left(1+\tfrac{\gamma_j}{3}\right)\\
&\quad +\frac{1}{n}\sum_{\substack{\mathcal{W}\subseteq\{1,\ldots,r\}\\ \mathcal{W}=\{i_1,\ldots,i_{|\mathcal{W}|}\}}}\sum_{\ell_{i_1}=1}^{p_{i_1}-1}\cdots\sum_{\ell_{i_{|\mathcal{W}|}}=1}^{p_{i_{|\mathcal{W}|}}-1}\prod_{j=1}^{s}\left(1+\gamma_j\left[B_2\left(\left\{\sum_{i\in\mathcal{W}}\frac{\ell_i z_{i,j}}{p_i}\right\}\right)+\frac{1}{3}\right]\right)\\
&= \left(1+\tfrac{\gamma_s}{3}\right)e_{n,s-1}^2(\boldsymbol{z}_1^{(s-1)},\ldots,\boldsymbol{z}_r^{(s-1)})\\
&\quad +\frac{\gamma_s}{n}\sum_{\substack{\mathcal{W}\subseteq\{1,\ldots,r\}\\ \mathcal{W}=\{i_1,\ldots,i_{|\mathcal{W}|}\}}}\sum_{\ell_{i_1}=1}^{p_{i_1}-1}\cdots\sum_{\ell_{i_{|\mathcal{W}|}}=1}^{p_{i_{|\mathcal{W}|}}-1}\left[\prod_{j=1}^{s-1}\left(1+\gamma_j\left[B_2\left(\left\{\sum_{i\in\mathcal{W}}\frac{\ell_i z_{i,j}}{p_i}\right\}\right)+\frac{1}{3}\right]\right)\right.\\
&\qquad\qquad\qquad\qquad\left.\times B_2\left(\left\{\sum_{i\in\mathcal{W}}\frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right].
\end{aligned}
$$

The result now follows from Lemma 1 below. □

The following lemma is a generalization of Lemma 2.2 in [2].

Lemma 1. *Suppose that $r\geq 1$, $p_1,\ldots,p_r$ are distinct prime numbers and $\mathcal{W}\subseteq\{1,\ldots,r\}$. For each $i\in\mathcal{W}$, suppose that ℓ_i satisfies $1\leq\ell_i\leq p_i-1$. Let*

$$S_r := B_2\left(\left\{\sum_{i\in\mathcal{W}}\frac{\ell_i z_{i,s}}{p_i}\right\}\right),$$

and for $m=1,\ldots,r-1$ define

$$S_m := \frac{1}{(p_{m+1}-1)\cdots(p_r-1)}\sum_{z_{m+1,s}=1}^{p_{m+1}-1}\cdots\sum_{z_{r,s}=1}^{p_r-1}B_2\left(\left\{\sum_{i\in\mathcal{W}}\frac{\ell_i z_{i,s}}{p_i}\right\}\right).$$

Then we have

$$
\begin{aligned}
S_m = &\prod_{i\in\mathcal{W}\setminus\{1,\ldots,m\}}(p_i-1)^{-1}\\
&\times\sum_{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,m\}}\left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,m\}|}}{\prod_{i\in\mathfrak{u}}p_i}B_2\left(\left\{\prod_{i\in\mathfrak{u}}p_i\sum_{i\in\mathcal{W}\cap\{1,\ldots,m\}}\frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right]
\end{aligned}
$$

for all $m=1,\ldots,r$.

Proof. We will prove this by a ‘backward’ induction on m. Clearly, the result holds when $m=r$. Suppose the result holds for some $m=t$, where $2\leq t\leq r$, that is,

$$S_t = \prod_{i\in\mathcal{W}\setminus\{1,\ldots,t\}} (p_i-1)^{-1} \times \sum_{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t\}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t\}|}}{\prod_{i\in\mathfrak{u}} p_i} B_2\left(\left\{ \prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t\}} \frac{\ell_i z_{i,s}}{p_i} \right\}\right)\right]. \quad (1.6)$$

By the definition of S_{t-1}, it is easy to see that

$$S_{t-1} = \frac{1}{p_t-1}\sum_{z_{t,s}=1}^{p_t-1} S_t.$$

We will show that the result also holds for $m=t-1$ by considering the cases $t\notin\mathcal{W}$ and $t\in\mathcal{W}$ separately.

If $t\notin\mathcal{W}$, we see from (1.6) that S_t does not depend on $z_{t,s}$ and thus $S_{t-1}=S_t$. Moreover, the right-hand side of (1.6) is unchanged if we replace t by $t-1$. Thus in this case the result holds for $m=t-1$.

Now we consider the case where $t\in\mathcal{W}$. To find the expression for S_{t-1}, we need to average (1.6) over $z_{t,s}$. More precisely, we need to average the B_2 term on the right-hand side of (1.6) over $z_{t,s}$. Using the property

$$B_2(x) = \frac{1}{2\pi^2}\sum_{h\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi \mathrm{i} h x}}{h^2}, \quad (1.7)$$

we can write

$$B_2\left(\left\{ \prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t\}} \frac{\ell_i z_{i,s}}{p_i} \right\}\right)$$
$$= \frac{1}{2\pi^2}\sum_{h\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{h^2} \prod_{k\in\mathcal{W}\cap\{1,\ldots,t-1\}} e^{2\pi \mathrm{i} h\left(\prod_{i\in\mathfrak{u}} p_i\right)\ell_k z_{k,s}/p_k} \cdot e^{2\pi \mathrm{i} h\left(\prod_{i\in\mathfrak{u}} p_i\right)\ell_t z_{t,s}/p_t} \right).$$

Here $e^{2\pi \mathrm{i} h\left(\prod_{i\in\mathfrak{u}} p_i\right)\ell_t z_{t,s}/p_t}$ is the only term that depends on $z_{t,s}$. As $t\notin\mathfrak{u}$ it can be shown that

$$\frac{1}{p_t-1}\sum_{z_{t,s}=1}^{p_t-1} e^{2\pi \mathrm{i} h\left(\prod_{i\in\mathfrak{u}} p_i\right)\ell_t z_{t,s}/p_t} = \begin{cases} 1, & \text{if } h \text{ is a multiple of } p_t,\\ -\frac{1}{p_t-1}, & \text{otherwise.}\end{cases}$$

Thus with some algebraic manipulations, we can obtain the average

$$\frac{1}{p_t-1}\sum_{z_{t,s}=1}^{p_t-1} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)$$
$$= \frac{1}{p_t(p_t-1)} B_2\left(\left\{\prod_{i\in\mathfrak{u}\cup\{t\}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)$$
$$- \frac{1}{p_t-1} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right),$$

which leads to

$$S_{t-1} = \prod_{i\in\mathcal{W}\setminus\{1,\ldots,t-1\}} (p_i-1)^{-1} F_{t-1},$$

where

$$F_{t-1}$$
$$= \sum_{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t\}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t\}|}}{\prod_{i\in\mathfrak{u}\cup\{t\}} p_i} B_2\left(\left\{\prod_{i\in\mathfrak{u}\cup\{t\}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right]$$
$$- \sum_{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t\}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t\}|}}{\prod_{i\in\mathfrak{u}} p_i} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right].$$

The two sums in F_{t-1} can be combined by changing the summation indices as follows:

$$F_{t-1}$$
$$= \sum_{\substack{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t-1\}\\ t\in\mathfrak{u}}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t-1\}|}}{\prod_{i\in\mathfrak{u}} p_i} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right]$$
$$+ \sum_{\substack{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t-1\}\\ t\notin\mathfrak{u}}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t-1\}|}}{\prod_{i\in\mathfrak{u}} p_i} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right]$$
$$= \sum_{\mathfrak{u}\subseteq\mathcal{W}\setminus\{1,\ldots,t-1\}} \left[\frac{(-1)^{|\mathfrak{u}|+|\mathcal{W}\setminus\{1,\ldots,t-1\}|}}{\prod_{i\in\mathfrak{u}} p_i} B_2\left(\left\{\prod_{i\in\mathfrak{u}} p_i \sum_{i\in\mathcal{W}\cap\{1,\ldots,t-1\}} \frac{\ell_i z_{i,s}}{p_i}\right\}\right)\right].$$

Thus the result holds for $m = t-1$ when $t \in \mathcal{W}$. Hence by induction, the result holds for all $m = 1, 2, \ldots, r$. This completes the proof. □

1.3 Error bounds

For any $n \geq 2$ let $\mathcal{Z}_n$ be given by $\{1 \leq a \leq n-1 : \gcd(a,n) = 1\}$ and let $\phi(n)$ be the number of elements in the set $\mathcal{Z}_n$. In [1] it was shown that for any $n \geq 2$ there exists a generating vector $\boldsymbol{v} = (v_1, \ldots, v_d)$ with $\gcd(v_j, n) = 1$ for $1 \leq j \leq d$ such that for all $1/2 < \lambda \leq 1$ the following bound holds:

$$e^2_{n,d}(\boldsymbol{v}) \leq \phi(n)^{-\frac{1}{\lambda}} \prod_{j=1}^{d} \left(\left(1 + \tfrac{\gamma_j}{3}\right)^{\lambda} + 2\zeta(2\lambda) \left(\tfrac{\gamma_j}{2\pi^2}\right)^{\lambda} \right)^{\frac{1}{\lambda}}, \tag{1.8}$$

where $\zeta(s) = \sum_{h=1}^{\infty} h^{-s}$ is the Riemann zeta function. It was also shown in [1] that such a vector $\boldsymbol{v}$ can be constructed by a component-by-component algorithm which searches over all elements in $\mathcal{Z}_n$ in each step. In Algorithm 1 we search only over $(p_1 - 1) + \cdots + (p_r - 1)$ values in each step, which is generally much smaller than $|\mathcal{Z}_n| = \phi(n) = (p_1 - 1) \cdots (p_r - 1)$. In Theorem 2 below we give an error bound for the worst-case error using lattice rules with generating vectors constructed by Algorithm 1. As the search cost is reduced in each step, it is not surprising that the error bound is not as good as (1.8).

We will use the following lemma where we write the square worst-case error as one sum.

Lemma 2. *Let $r \geq 1$, $p_1, \ldots, p_r$ be distinct prime numbers and $n = p_1 \cdots p_r$. We can write*

$$e^2_{n,d}(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r) = \sum_{\boldsymbol{h} \in \mathcal{L}_d} \prod_{j=1}^{d} \varphi\left(2, 1 + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right),$$

where $\mathcal{L}_d = \{\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\} : \boldsymbol{h} \cdot \boldsymbol{z}_m \equiv 0 \pmod{p_m}$ for each $m = 1, \ldots, r\}$ and

$$\varphi(\alpha, \beta, \gamma, h) = \begin{cases} \beta, & \text{if } h = 0, \\ \gamma h^{-\alpha}, & \text{if } h \neq 0. \end{cases}$$

Proof. We use (1.7) to rewrite the square worst-case error (1.3) as follows:

$$
\begin{aligned}
&e^2_{n,d}(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_r)\\
&= -\prod_{j=1}^d \left(1+\tfrac{\gamma_j}{3}\right)\\
&\quad + \frac{1}{n}\sum_{\ell_1=1}^{p_1}\cdots\sum_{\ell_r=1}^{p_r}\prod_{j=1}^d \left(1+\frac{\gamma_j}{3}+\frac{\gamma_j}{2\pi^2}\sum_{h\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi \mathrm{i} h(\ell_1 z_{1,j}/p_1+\cdots+\ell_r z_{r,j}/p_r)}}{h^2}\right)\\
&= -\prod_{j=1}^d \left(1+\tfrac{\gamma_j}{3}\right)\\
&\quad + \frac{1}{n}\sum_{\ell_1=1}^{p_1}\cdots\sum_{\ell_r=1}^{p_r}\prod_{j=1}^d \left(\sum_{h\in\mathbb{Z}}\varphi\left(2,1+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h_j\right)e^{2\pi \mathrm{i} h(\ell_1 z_{1,j}/p_1+\cdots+\ell_r z_{r,j}/p_r)}\right)\\
&= \sum_{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}\left[\prod_{i=1}^r\left(\frac{1}{p_i}\sum_{\ell_i=1}^{p_i}e^{2\pi \mathrm{i}\ell_i \boldsymbol{h}\cdot\boldsymbol{z}_i/p_i}\right)\prod_{j=1}^d\varphi\left(2,1+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h_j\right)\right].
\end{aligned}
$$

The result now follows from the fact that for each i satisfying $1\le i\le r$, $\sum_{\ell_i=1}^{p_i}e^{2\pi \mathrm{i}\ell_i\boldsymbol{h}\cdot\boldsymbol{z}_i/p_i}$ is p_i if $\boldsymbol{h}\cdot\boldsymbol{z}_i$ is a multiple of p_i and 0 otherwise. □

Theorem 2 below shows that the lattice rule with generating vector constructed by Algorithm 1 achieves a rate of convergence $O(p_1^{-1+\delta}p_2^{-1/2}\cdots p_r^{-1/2})$ for $\delta>0$. Note that this result is a generalization of Theorem 3.3 in [2] (and also Theorem 7 in [1]). The proof of Theorem 2 makes use of Jensen's inequality (see [3]), which states that for $\{a_i\}$ a sequence of positive numbers,

$$\left(\sum a_i\right)^\lambda \le \sum a_i^\lambda \quad \text{for } 0<\lambda\le 1.$$

Theorem 2. *Let $r\ge 1$, $p_1,\ldots,p_r$ be distinct prime numbers, $n=p_1\cdots p_r$, and let the d-dimensional vectors $\hat{\boldsymbol{z}}_1,\ldots,\hat{\boldsymbol{z}}_r$ be constructed using Algorithm 1. Then for each $s=1,\ldots,d$ we have*

$$e^2_{n,s}(\hat{\boldsymbol{z}}_1^{(s)},\ldots,\hat{\boldsymbol{z}}_r^{(s)})\le (p_1-1)^{-\frac{1}{\lambda}}\prod_{i=2}^r(p_i-1)^{-1}\prod_{j=1}^s\left(\left(1+\tfrac{\gamma_j}{3}\right)^\lambda+2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda\right)^{\frac{1}{\lambda}}$$

for all $\frac{1}{2}<\lambda\le 1$.

Proof. We prove the result by induction. For $s=1$, we have

$$e^2_{n,1}(\hat{\boldsymbol{z}}_1^{(1)},\ldots,\hat{\boldsymbol{z}}_r^{(1)})=e^2_{n,1}(1,\ldots,1)=\sum_{h\in\mathcal{L}_1}\varphi\left(2,1+\tfrac{\gamma_1}{3},\tfrac{\gamma_1}{2\pi^2},h\right),$$

where we sum over those $h\neq 0$ for which $h\equiv 0\,(\mathrm{mod}\,p_m)$ for each $m=1,\ldots,r$. In other words, we sum only over those $h\neq 0$ such that $n|h$. Thus

$$e_{n,1}^2(\hat{\boldsymbol{z}}_1^{(1)},\dots,\hat{\boldsymbol{z}}_r^{(1)}) = \frac{\gamma_1}{2\pi^2}\sum_{h\in\mathcal{L}_1} h^{-2} = \frac{\gamma_1}{n^2\pi^2}\sum_{h=1}^{\infty} h^{-2} = \frac{\gamma_1}{6n^2},$$

where we used that $\sum_{h=1}^{\infty} h^{-2} = \pi^2/6$. It is not difficult to show that this expression satisfies the required bound with $s = 1$. Therefore the result is true for $s = 1$.

For s satisfying $2 \le s \le d$, suppose the vectors $\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)}$ have already been constructed by Algorithm 1 and they satisfy

$$\begin{aligned} &e_{n,s-1}^2(\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)}) \\ &\le (p_1-1)^{-\frac{1}{\lambda}}\prod_{i=2}^{r}(p_i-1)^{-1}\prod_{j=1}^{s-1}\left(\left(1+\tfrac{\gamma_j}{3}\right)^{\lambda} + 2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \end{aligned} \tag{1.9}$$

for all $\frac{1}{2} < \lambda \le 1$. For each $m = 2\dots,r$, it follows from the definition of $\Theta_{n,s}^{(m)}$, given by (1.4) and (1.5), that $\Theta_{n,s}^{(m-1)}$ is the average of $\Theta_{n,s}^{(m)}$ over all values of $z_{m,s}$. Using this, it is not hard to see from Step 2 of Algorithm 1 that $\hat{z}_{1,s},\dots,\hat{z}_{r,s}$ will satisfy

$$\begin{aligned} e_{n,s}^2(\hat{\boldsymbol{z}}_1^{(s)},\dots,\hat{\boldsymbol{z}}_r^{(s)}) &= \Theta_{n,s}^{(r)}(\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)};\hat{z}_{1,s},\dots,\hat{z}_{r,s}) \\ &\le \Theta_{n,s}^{(r-1)}(\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)};\hat{z}_{1,s}\dots,\hat{z}_{r-1,s}) \\ &\le \dots \le \Theta_{n,s}^{(1)}(\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)};\hat{z}_{1,s}). \end{aligned}$$

Hence the result is proved if we can prove that $\Theta_{n,s}^{(1)}(\hat{\boldsymbol{z}}_1^{(s-1)},\dots,\hat{\boldsymbol{z}}_r^{(s-1)};\hat{z}_{1,s})$ satisfies the required bound.

From Lemma 2 we have

$$e_{n,s}^2(\boldsymbol{z}_1^{(s)},\dots,\boldsymbol{z}_r^{(s)}) = \sum_{\boldsymbol{h}\in\mathcal{L}_s}\prod_{j=1}^{s}\varphi\left(2, 1+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h_j\right),$$

where $\mathcal{L}_s = \{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\} : \boldsymbol{h}\cdot\boldsymbol{z}_m \equiv 0 \pmod{p_m} \text{ for each } m = 1,\dots,r\}$. For $h_s = 0$ in the last sum above, we obtain

$$\left(1+\tfrac{\gamma_s}{3}\right) e_{n,s-1}^2(\boldsymbol{z}_1^{(s-1)},\dots,\boldsymbol{z}_r^{(s-1)})$$

and for $n|h_s$ with $h_s \neq 0$ we get

$$\frac{\gamma_s}{6n^2}\left(e_{n,s-1}^2(\boldsymbol{z}_1^{(s-1)},\dots,\boldsymbol{z}_r^{(s-1)}) + \prod_{j=1}^{s-1}\left(1+\tfrac{\gamma_j}{3}\right)\right).$$

Therefore we have

$$
\begin{aligned}
&e^2_{n,s}((\hat{\boldsymbol{z}}_1^{(s-1)}, z_{1,s}), \ldots, (\hat{\boldsymbol{z}}_r^{(s-1)}, z_{r,s})) \\
&= \left(1 + \tfrac{\gamma_s}{3}\right) e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}) \\
&\quad + \frac{\gamma_s}{6n^2}\left(e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}) + \prod_{j=1}^{s-1}\left(1 + \tfrac{\gamma_j}{3}\right)\right) \\
&\quad + \gamma_s \kappa(z_{1,s}, \ldots, z_{r,s}),
\end{aligned}
$$

with

$$
\kappa(z_{1,s}, \ldots, z_{r,s}) := \gamma_s^{-1} \sum_{\boldsymbol{h} \in \bar{\mathcal{L}}_s} \prod_{j=1}^{s} \varphi\left(2, 1 + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right),
$$

where $\bar{\mathcal{L}}_s = \{\boldsymbol{h} \in \mathbb{Z}^s : n \nmid h_s \text{ and } \boldsymbol{h} \cdot (\hat{\boldsymbol{z}}_m^{(s-1)}, z_{m,s}) \equiv 0 \,(\bmod\ p_m) \text{ for each } m = 1, \ldots, r\}$. Using this expression for the worst-case error we obtain

$$
\begin{aligned}
&\Theta^{(1)}_{n,s}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}; z_{1,s}) \\
&= \left(1 + \tfrac{\gamma_s}{3}\right) e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}) \\
&\quad + \frac{\gamma_s}{6n^2}\left(e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}) + \prod_{j=1}^{s-1}\left(1 + \tfrac{\gamma_j}{3}\right)\right) \\
&\quad + \gamma_s \Phi_{n,s}(z_{1,s}),
\end{aligned} \tag{1.10}
$$

where

$$
\Phi_{n,s}(z_{1,s}) := \frac{1}{(p_2 - 1)\cdots(p_r - 1)} \sum_{z_{2,s}=1}^{p_2-1} \cdots \sum_{z_{r,s}=1}^{p_r-1} \kappa(z_{1,s}, \ldots, z_{r,s}). \tag{1.11}
$$

In the following let $1/2 < \lambda \le 1$. As $\hat{z}_{1,s}$ is chosen by Algorithm 1 such that $\Theta^{(1)}_{n,s}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}; \hat{z}_{1,s}) \le \Theta^{(1)}_{n,s}(\hat{\boldsymbol{z}}_1^{(s-1)}, \ldots, \hat{\boldsymbol{z}}_r^{(s-1)}; z_{1,s})$ for all $1 \le z_{1,s} \le p_1 - 1$, it follows that $\Phi_{n,s}(\hat{z}_{1,s}) \le \Phi_{n,s}(z_{1,s})$ for all $1 \le z_{1,s} \le p_1 - 1$ and therefore we also have $(\Phi_{n,s}(\hat{z}_{1,s}))^\lambda \le (\Phi_{n,s}(z_{1,s}))^\lambda$ for all $1 \le z_{1,s} \le p_1 - 1$. Now we use (1.11) and Jensen's inequality to obtain

$$
\begin{aligned}
(\Phi_{n,s}(\hat{z}_{1,s}))^\lambda &\le \frac{1}{p_1 - 1} \sum_{z_{1,s}=1}^{p_1-1} (\Phi_{n,s}(z_{1,s}))^\lambda \\
&\le \frac{1}{(p_1 - 1)(p_2 - 1)^\lambda \cdots (p_r - 1)^\lambda} \sum_{z_{1,s}=1}^{p_1-1} \cdots \sum_{z_{r,s}=1}^{p_r-1} (\kappa(z_{1,s}, \ldots, z_{r,s}))^\lambda.
\end{aligned}
$$

By (1.2) we have for $\boldsymbol{h}' \in \mathbb{Z}^s$ that

$$
\boldsymbol{h}' \cdot (\hat{\boldsymbol{v}}^{(s-1)}, v_s) \equiv \boldsymbol{h}' \cdot (\hat{\boldsymbol{z}}_1^{(s-1)}, z_{1,s}) \frac{n}{p_1} + \cdots + \boldsymbol{h}' \cdot (\hat{\boldsymbol{z}}_r^{(s-1)}, z_{r,s}) \frac{n}{p_r} \,(\bmod\, n)
$$

and therefore the condition $\boldsymbol{h}' = (\boldsymbol{h}, h_s) \in \bar{\mathcal{L}}_s$ is equivalent to $\boldsymbol{h} \cdot \hat{\boldsymbol{v}}^{(s-1)} \equiv -h_s v_s \pmod n$. Thus we have

$$\begin{aligned}
&\kappa(v_s) := \kappa(z_{1,s}, \ldots, z_{r,s}) \\
&= \gamma_s^{-1} \sum_{\boldsymbol{h}' \in \bar{\mathcal{L}}_s} \prod_{j=1}^{s} \varphi\left(2, 1 + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right) \\
&= \gamma_s^{-1} \sum_{\substack{h_s \in \mathbb{Z} \\ n \nmid h_s}} \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^{s-1} \\ \boldsymbol{h} \cdot \hat{\boldsymbol{v}}^{(s-1)} \equiv -h_s v_s \pmod n}} \prod_{j=1}^{s} \varphi\left(2, 1 + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right) \\
&= \frac{1}{2\pi^2} \sum_{\substack{h_s \in \mathbb{Z} \\ n \nmid h_s}} \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^{s-1} \\ \boldsymbol{h} \cdot \hat{\boldsymbol{v}}^{(s-1)} \equiv -h_s v_s \pmod n}} h_s^{-2} \prod_{j=1}^{s-1} \varphi\left(2, 1 + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right).
\end{aligned}$$

From the proof of Lemma 3 in [1] we have

$$\begin{aligned}
&\sum_{v_s \in \mathcal{Z}_n} (\kappa(v_s))^\lambda \\
&\le 2\zeta(2\lambda)(1 - n^{-2\lambda}) \left(\prod_{j=1}^{s-1} \left(\left(1 + \tfrac{\gamma_j}{3}\right)^\lambda + 2\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda \zeta(2\lambda) \right) - \prod_{j=1}^{s-1} \left(1 + \tfrac{\gamma_j}{3}\right)^\lambda \right),
\end{aligned}$$

and since

$$\sum_{z_{1,s}=1}^{p_1 - 1} \cdots \sum_{z_{r,s}=1}^{p_r - 1} (\kappa(z_{1,s}, \ldots, z_{r,s}))^\lambda = \sum_{v_s \in \mathcal{Z}_n} (\kappa(v_s))^\lambda,$$

we obtain

$$\begin{aligned}
(\varPhi_{n,s}(\hat{z}_{1,s}))^\lambda &\le \frac{1}{(p_1 - 1)(p_2 - 1)^\lambda \cdots (p_r - 1)^\lambda} \sum_{v_s \in \mathcal{Z}_n} (\kappa(v_s))^\lambda \\
&\le \frac{2\zeta(2\lambda)(1 - n^{-2\lambda})}{(p_1 - 1)(p_2 - 1)^\lambda \cdots (p_r - 1)^\lambda} \qquad (1.12) \\
&\quad \times \left(\prod_{j=1}^{s-1} \left(\left(1 + \tfrac{\gamma_j}{3}\right)^\lambda + 2\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda \zeta(2\lambda) \right) - \prod_{j=1}^{s-1} \left(1 + \tfrac{\gamma_j}{3}\right)^\lambda \right).
\end{aligned}$$

By using Jensen's inequality we obtain

$$\left(n^{-2}e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)},\ldots,\hat{\boldsymbol{z}}_r^{(s-1)}) + n^{-2}\prod_{j=1}^{s-1}\left(1+\tfrac{\gamma_j}{3}\right) + 6\Phi_{n,s}(\hat{z}_{1,s})\right)^{\lambda}$$
$$\le n^{-2\lambda}(e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)},\ldots,\hat{\boldsymbol{z}}_r^{(s-1)}))^{\lambda} + n^{-2\lambda}\prod_{j=1}^{s-1}\left(1+\tfrac{\gamma_j}{3}\right)^{\lambda} + 6^{\lambda}(\Phi_{n,s}(\hat{z}_{1,s}))^{\lambda}$$
$$\le 2\zeta(2\lambda)\left(\tfrac{3}{\pi^2}\right)^{\lambda}(p_1-1)^{-1}\prod_{i=2}^{r}(p_i-1)^{-\lambda}\prod_{j=1}^{s-1}\left(\left(1+\tfrac{\gamma_j}{3}\right)^{\lambda} + 2\left(\tfrac{\gamma_j}{2\pi^2}\right)^{\lambda}\zeta(2\lambda)\right),$$

where the last inequality follows from (1.9) and (1.12). Now we use (1.9), (1.10) and the inequality above to obtain

$$\Theta^{(1)}_{n,s}(\hat{\boldsymbol{z}}_1^{(s-1)},\ldots,\hat{\boldsymbol{z}}_r^{(s-1)};\hat{z}_{1,s})$$
$$= \left(1+\tfrac{\gamma_s}{3}\right)e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)},\ldots,\hat{\boldsymbol{z}}_r^{(s-1)})$$
$$+\frac{\gamma_s}{6}\left(n^{-2}e^2_{n,s-1}(\hat{\boldsymbol{z}}_1^{(s-1)},\ldots,\hat{\boldsymbol{z}}_r^{(s-1)}) + n^{-2}\prod_{j=1}^{s-1}\left(1+\tfrac{\gamma_j}{3}\right) + 6\Phi_{n,s}(\hat{z}_{1,s})\right)$$
$$\le (p_1-1)^{-\frac{1}{\lambda}}\prod_{i=2}^{r}(p_i-1)^{-1}\prod_{j=1}^{s}\left(\left(1+\tfrac{\gamma_j}{3}\right)^{\lambda} + 2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}.$$

This completes the proof. □

1.4 Numerical experiments

For n a product of r distinct primes, the construction cost of the Partial Search algorithm is $O(n(p_1+\cdots+p_r)d^2)$ operations, which is minimized when each prime p_m is roughly $n^{1/r}$. Based on our theoretical rate of convergence $O(p_1^{-1+\delta}p_2^{-1/2}\cdots p_r^{-1/2})$ for $\delta>0$, it would seem intuitive to take the first prime p_1 in the decomposition of n to be much larger than the rest of the primes. However, the numerical experiments in [2], with a two-prime decomposition, have indicated that the observed rate of convergence does not depend on how the two primes are chosen. Therefore we will take the prime numbers to be roughly equal in our numerical experiments here, since in this case the construction cost is minimized. Table 1.1 below shows a comparison of the construction cost against the theoretical rate of convergence. It is clear from this table that as r increases from 2 to 5, the construction cost reduces considerably (for $n=10^6$ the reduction from $r=2$ to $r=5$ is by a factor of 63), but at the same time the theoretical rate of convergence is compromised. The aim of our numerical experiments here is to see how the worst-case errors and the observed rate of convergence behave when more primes are used in the decomposition of n.

Table 1.1 Partial Search analysis

r	$n = p_1 p_2 \cdots p_r$	Cost of Construction	Rate of Convergence
2	$p_m \approx n^{1/2}$	$O(n^{1.5}d^2)$	$O(n^{-0.75+\delta})$
3	$p_m \approx n^{1/3}$	$O(n^{1.33}d^2)$	$O(n^{-0.667+\delta})$
4	$p_m \approx n^{1/4}$	$O(n^{1.25}d^2)$	$O(n^{-0.625+\delta})$
5	$p_m \approx n^{1/5}$	$O(n^{1.2}d^2)$	$O(n^{-0.6+\delta})$

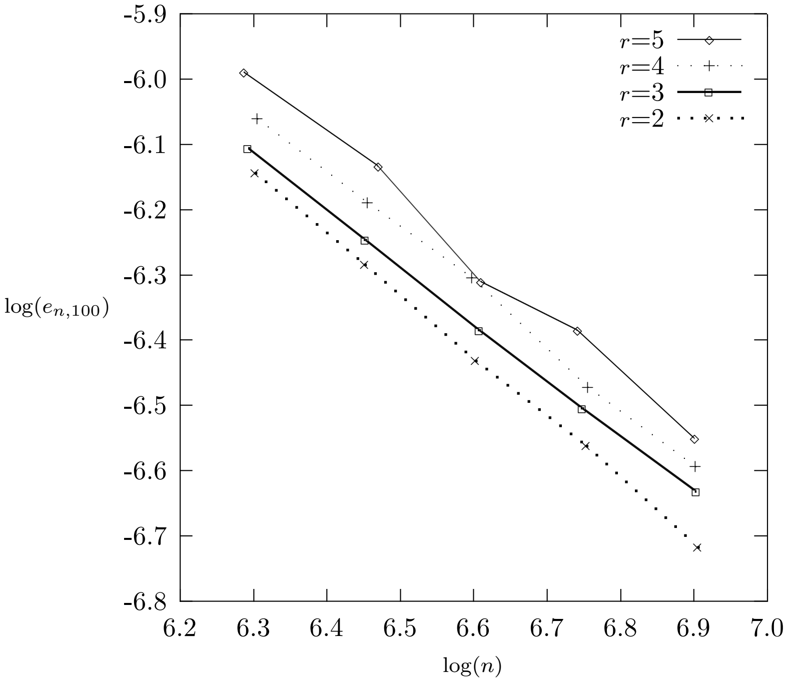
For n roughly 2 million, 2.8 million, 4 million, 5.7 million, and 8 million, we decompose n into a product of 2, 3, 4 and 5 primes. We choose $d = 100$ and consider two choices of weights: $\gamma_j = 0.5^j$ and $\gamma_j = 1/j^2$, and we compare the worst-case errors of the randomly shifted rank-1 lattice rules constructed via Partial Search. The prime numbers we use and the numerical results are presented in Tables 1.2 to 1.5 in the appendix.

To give a better picture of what the numbers mean, we plot these into two graphs: Figure 1.1 shows a graph of $\log(e_{n,100})$ against $\log(n)$ with $\gamma_j = 0.5^j$, and Figure 1.2 shows the same information for the case where $\gamma_j = 1/j^2$.

It can be seen from Figure 1.1 and Figure 1.2 that the rate of convergence in our examples remains roughly constant, regardless of how many primes we use. By increasing the number of primes by one, the worst-case error seems to increase by a certain factor (for our examples, we see from the tables in the appendix that this factor is smaller than 1.25; this is also true for the numerical results in [2], where more choices of weights were considered). Based on these considerations, the construction cost for lattice rules with a worst-case error lower than some fixed constant c, is minimized in most cases (depending on c and the observed rate of convergence) by choosing r between 2 and 5 (with n growing bigger as r increases). Further, in Figure 1.1 we see that choosing many primes may produce some outliers, which might become more extreme as the number of prime factors increases. This may be due to the fact that the prime numbers in this case are quite small. For $r = 5$ in our examples, the prime numbers we used are at most 37, even for n about 8 million.

As mentioned earlier, the reduction in the construction cost is substantial, especially for very large n, whereas the increase of the worst-case error is mild. Therefore, using n as a product of up to 5 primes appears to be useful in appropriate cases.

Fig. 1.1 Partial Search with $\gamma_j = 0.5^j$



Appendix: Tables of numerical results

Table 1.2 Partial Search with $r = 2$

n	p_1	p_2	$e_{n,100}$ with $\gamma_j = 0.5^j$	$e_{n,100}$ with $\gamma_j = 1/j^2$
2005007	1423	1409	7.1750e-07	1.9173e-06
2825617	1693	1669	5.1953e-07	1.4570e-06
4003997	2003	1999	3.7002e-07	1.0686e-06
5659637	2381	2377	2.7406e-07	8.0221e-07
8037211	2837	2833	1.9148e-07	5.9812e-07

Fig. 1.2 Partial search with $\gamma_j = 1/j^2$

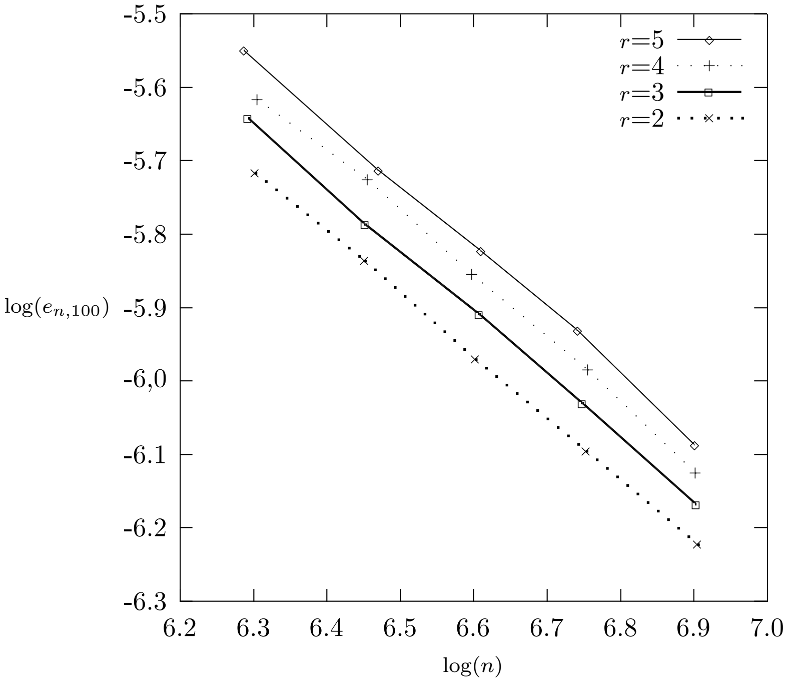


Table 1.3 Partial Search with $r = 3$

n	p_1	p_2	p_3	$e_{n,100}$ with $\gamma_j = 0.5^j$	$e_{n,100}$ with $\gamma_j = 1/j^2$
1966087	137	127	113	7.8342e-07	2.2806e-06
2837407	149	139	137	5.6658e-07	1.6320e-06
4055929	167	163	149	4.1256e-07	1.2326e-06
5605027	181	179	173	3.1262e-07	9.3287e-07
8022431	211	197	193	2.3335e-07	6.7881e-07

Table 1.4 Partial Search with $r = 4$

n	p_1	p_2	p_3	p_4	$e_{n,100}$ with $\gamma_j = 0.5^j$	$e_{n,100}$ with $\gamma_j = 1/j^2$
2022161	43	41	37	31	8.6847e-07	2.4180e-06
2857177	53	47	37	31	6.4611e-07	1.8787e-06
3963181	53	47	43	37	4.9601e-07	1.3965e-06
5699779	61	53	43	41	3.3709e-07	1.0358e-06
7989013	67	59	47	43	2.5473e-07	7.4932e-07

Table 1.5 Partial Search with $r = 5$

n	p_1	p_2	p_3	p_4	p_5	$e_{n,100}$ with $\gamma_j = 0.5^j$	$e_{n,100}$ with $\gamma_j = 1/j^2$
1937221	31	23	19	13	11	1.0260e-06	2.8180e-06
2956811	31	29	23	13	11	7.3529e-07	1.9358e-06
4075291	37	31	19	17	11	4.8902e-07	1.5027e-06
5513629	37	31	23	19	11	4.1240e-07	1.1734e-06
7971317	37	29	23	19	17	2.8110e-07	8.1762e-07

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