On quasi-Monte Carlo rules achieving higher order convergence

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Summary. Quasi-Monte Carlo rules which can achieve arbitrary high order of convergence have been introduced recently. The construction is based on digital nets and the analysis of the integration error uses Walsh functions. Various approaches have been used to show arbitrary high convergence. In this paper we explain the ideas behind higher order quasi-Monte Carlo rules by leaving out most of the technical details and focusing on the ideas behind it.

1 Introduction

In this paper we study the approximation of multivariate integrals of the form

$$\int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

by quasi-Monte Carlo rules

$$\frac{1}{N}\sum_{h=0}^{N-1} f(\boldsymbol{x}_h).$$

Whereas the classical theory, see [9, 10, 11], focused on functions with bounded variation (or functions with square integrable partial mixed derivatives up to first order in each variable) or periodic functions, see [16], here we focus on functions which are not periodic and are smooth. The smoothness is a requirement if one wants to achieve convergence rates of order $N^{-\alpha}(\log N)^{c(s,\alpha)}$ with $\alpha > 1$ (here $c(s,\alpha)$ is a function which depends only on the dimension s and the smoothness α), as, for example, by the lower bound by Sharygin [15] we can in general at most get $N^{-1}(\log N)^s$ for functions which have only bounded variation but no additional smoothness.

So let us assume our integrand $f : [0,1]^s \to \mathbb{R}$ is smooth. For $s = 1$ we consider the norm

$$\|f\|_\alpha = \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 + \cdots + \left(\int_0^1 f^{(\alpha-1)}(x)\,\mathrm{d}x\right)^2 + \int_0^1 |f^{(\alpha)}(x)|^2\,\mathrm{d}x,$$

and the corresponding inner product

$$\langle f, g\rangle_\alpha = \int_0^1 f(x)\,\mathrm{d}x \int_0^1 g(x)\,\mathrm{d}x + \cdots + \int_0^1 f^{(\alpha-1)}(x)\,\mathrm{d}x \int_0^1 g^{(\alpha-1)}(x)\,\mathrm{d}x + \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x)\,\mathrm{d}x,$$

where $f^{(\tau)}$ denotes the τth derivative of f for $1 \le \tau \le \alpha$ and where $f^{(0)} = f$.

In dimensions $s > 1$ we consider the tensor product, but before we can do so we need some additional notation. Let $S = \{1, \ldots, s\}$, $\boldsymbol{x} = (x_1, \ldots, x_s)$ and for $u \subseteq S$ let $\boldsymbol{x}_u = (x_j)_{j \in u}$ denote the vector which only consists of the components x_j of $\boldsymbol{x}$ for which $j \in u$. Further, for $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_s) \in \{0, \ldots, \alpha\}^s$ let $|\boldsymbol{\tau}| = \tau_1 + \cdots + \tau_s$, $f^{(\boldsymbol{\tau})}(\boldsymbol{x}) = \frac{\partial^{|\boldsymbol{\tau}|} f}{\partial x_1^{\tau_1} \cdots \partial x_s^{\tau_s}}(\boldsymbol{x})$ and for $\boldsymbol{\tau} = \boldsymbol{0}$ let $f^{(\boldsymbol{0})}(\boldsymbol{x}) = f(\boldsymbol{x})$.

We define a norm

$$\|f\|_\alpha = \sum_{u \subseteq \{1,\ldots,s\}} \sum_{\boldsymbol{\tau}_{S\setminus u} \subseteq \{0,\ldots,\alpha-1\}^{s-|u|}} \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{s-|u|}} f^{(\boldsymbol{\tau}_{S\setminus u}, \boldsymbol{\alpha}_u)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{S\setminus u}\right)^2 \mathrm{d}\boldsymbol{x}_u,$$

where $\boldsymbol{\tau}_{S\setminus u} \in \{0, \ldots, \alpha-1\}^{s-|u|}$ shall denote a vector for which τ_j does not occur for $j \in u$ and otherwise has a value in $\{0, \ldots, \alpha-1\}$, and where $(\boldsymbol{\tau}_{S\setminus u}, \boldsymbol{\alpha}_u)$ is the vector for which the jth component is α for $j \in u$ and τ_j for $j \in S \setminus u$. The corresponding inner product is given by

$$\langle f, g\rangle_\alpha = \sum_{u \subseteq \{1,\ldots,s\}} \sum_{\boldsymbol{\tau}_{S\setminus u} \subseteq \{0,\ldots,\alpha-1\}^{s-|u|}} \int_{[0,1]^{|u|}} \int_{[0,1]^{s-|u|}} f^{(\boldsymbol{\tau}_{S\setminus u}, \boldsymbol{\alpha}_u)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{S\setminus u} \int_{[0,1]^{s-|u|}} g^{(\boldsymbol{\tau}_{S\setminus u}, \boldsymbol{\alpha}_u)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{S\setminus u}\,\mathrm{d}\boldsymbol{x}_u.$$

We say that a function f has smoothness α if $\|f\|_\alpha < \infty$. In the papers on higher order quasi-Monte Carlo rules various definitions of smoothness have been used, different from the one just introduced, for technical reasons: In [4] the author considered a Korobov space of periodic functions for which the kth Fourier coefficient is of order $|k|^{-\alpha}$, i.e., functions in this space have $\|f\|_\alpha < \infty$, but are in addition also periodic. Non-periodic functions were first included in [5], but the results therein were based on a somewhat different norm purely for technical reasons. The results in [5] also include fractional smoothness, i.e., therein $\alpha > 1$ is allowed to be any real number. The function space considered in [5] was based on Walsh series, and it was shown that this space includes all smooth functions, i.e., functions with smoothness $\alpha > 1$. Later, it was shown in [6] that functions f with $\|f\|_\alpha < \infty$ are contained in this Walsh space. A function space with norm as above was finally considered in [1].

First results on convergence rates faster than $N^{-1}(\log N)^s$ were obtained in [14], where convergence rates of $N^{-3/2}(\log N)^{(s-1)/2}$ were shown using scrambled digital nets and in [3], where a convergence of $N^{-2+\delta}$, $\delta > 0$, was shown, also using randomized point sets.

There are two main hurdles to arrive at quasi-Monte Carlo rules which achieve the optimal order of convergence for functions with smoothness α, where $\alpha \in \mathbb{N}$ can be arbitrarily high.

The first main step towards proving higher order convergence of the integration error (i.e., convergence of $N^{-\alpha}(\log N)^{\alpha s}$ for any $\alpha \geq 1$) is a result concerning the decay of the Walsh coefficients. We will explain the details in Section 4.1. It requires a result on the decay of the Walsh coefficients of smooth functions, first shown explicitly in [5], see also [6].

The second main step is to construct point sets explicitly which can be used in a quasi-Monte Carlo rule. The construction scheme uses digital nets and a quality criterion on the generating matrices of such point sets can be obtained using the result in the first step. The details of this will be explained in Section 4.3.

It is useful to first look at how lattice rules can achieve arbitrary high order of convergence for smooth periodic functions, as part of the theory for non-periodic functions is similar, albeit much more technical.

2 Higher order convergence for smooth periodic functions using lattice rules

In this section we consider numerical integration using lattice rules, which will give us a basic understanding of how the theory on numerical integration works, see also [16] for a particular nice introduction to this theory.

2.1 Lattice rules

First let us introduce lattice rules. Assume we want a quasi-Monte Carlo rule with N points. For a real number x let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Then choose a vector $\boldsymbol{g} \in \{1, \ldots, N-1\}$ and use the quadrature rule

$$\frac{1}{N} \sum_{\ell=0}^{N-1} f\left(\left\{\frac{\ell \boldsymbol{g}}{N}\right\}\right).$$

This quadrature rule is called lattice rule.

Such rules work well with periodic functions. Before we can introduce the error analysis we need some understanding of the connection between smooth periodic functions and the decay of the Fourier coefficients.

2.2 Decay of the Fourier coefficients of smooth periodic functions

Let now $f : [0,1]^s \to \mathbb{R}$ be a smooth periodic function. I.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^s$ (here $\{0,1\}$ is the set consisting of the two elements 0 and 1), and any $\boldsymbol{\tau} \in \{0, \ldots, \alpha-1\}^s$ we have $f^{(\boldsymbol{\tau})}(\boldsymbol{x}) = f^{(\boldsymbol{\tau})}(\boldsymbol{y})$. Assume f has square integrable partial mixed derivatives up to order α in each variable, then $\|f\|_\alpha < \infty$. We assume in the following that $\alpha \ge 1$. Let the Fourier series of f be given by

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^s} \widehat{f}(\boldsymbol{k}) \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}},$$

where $\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + \cdots + k_s x_s$ and $\widehat{f}(\boldsymbol{k})$ is the $\boldsymbol{k}$th Fourier coefficient $\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x}) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}} \,\mathrm{d}\boldsymbol{x}$.

Consider the case $s = 1$ for a moment: Then $f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) \mathrm{e}^{2\pi \mathrm{i} k x}$. Assume that f is differentiable and let $\widehat{f'}(k)$ denote the kth Fourier coefficient of f', i.e., $\widehat{f'}(k) = \int_0^1 f'(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \,\mathrm{d}x$. Then by differentiating the Fourier series for f we obtain $2\pi \mathrm{i} k \widehat{f}(k) = \widehat{f'}(k)$, or, for $k \neq 0$, $\widehat{f}(k) = \widehat{f'}(k)/(2\pi \mathrm{i} k)$. Another way of obtaining the last formula for $k \neq 0$ is by using integration by parts:

$$\begin{aligned} \widehat{f}(k) &= \int_0^1 f(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \,\mathrm{d}x \\ &= -\frac{1}{2\pi \mathrm{i} k} [f(x) \mathrm{e}^{-2\pi \mathrm{i} k x}]_{x=0}^1 + \frac{1}{2\pi \mathrm{i} k} \int_0^1 f'(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \,\mathrm{d}x \\ &= \frac{\widehat{f'}(k)}{2\pi \mathrm{i} k}, \end{aligned}$$

as $f(0) = f(1)$. If, say, $\int_0^1 |f'(x)| \,\mathrm{d}x < \infty$, then the equation above implies that for $k \neq 0$ we have

$$|\widehat{f}(k)| = \frac{1}{2\pi |k|} \left| \int_0^1 f'(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \,\mathrm{d}x \right| \le \frac{1}{2\pi |k|} \int_0^1 |f'(x)| \,\mathrm{d}x.$$

Repeated use of the argument above shows that if f is α times differentiable, then $|\widehat{f}(k)| = \mathcal{O}(|k|^{-\alpha})$.

The case $s > 1$ works similarly. We have $|\widehat{f}(\boldsymbol{k})| = \mathcal{O}(|\bar{k}_1 \cdots \bar{k}_s|^{-\alpha})$, where $\bar{k} = k$ for $k \neq 0$ and 1 otherwise. The constant in the bound on the Fourier coefficient depends on the norm of the function, indeed, one can show that $|\widehat{f}(\boldsymbol{k})| \le C_{\alpha,s} |\bar{k}_1 \cdots \bar{k}_s|^{-\alpha} \|f\|_\alpha$ with some constant $C_{\alpha,s}$ independent of $\boldsymbol{k}$ and f.

2.3 Numerical integration

The following property is useful in analyzing the integration error of Fourier series when one approximates the integral with a lattice rule (we assume N is a prime number):

$$\frac{1}{N}\sum_{\ell=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \ell \boldsymbol{k}\cdot\boldsymbol{g}/N} = \begin{cases} 1 & \text{if } \boldsymbol{k}\cdot\boldsymbol{g} \equiv 0 \pmod N, \\ 0 & \text{otherwise.}\end{cases}$$

The set of all $\boldsymbol{k} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ for which the above sum is 1 is called the dual lattice, i.e., we have

$$\mathcal{L} = \{\boldsymbol{k} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\} : \boldsymbol{k}\cdot\boldsymbol{g} \equiv 0 \pmod N\}.$$

Using the Fourier series expansion of the function f we obtain

$$\begin{aligned}\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{\ell=0}^{N-1} f(\{\ell\boldsymbol{g}/N\})\right| &= \left|\widehat{f}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{Z}^s} \widehat{f}(\boldsymbol{k}) \frac{1}{N}\sum_{\ell=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\ell\boldsymbol{k}\cdot\boldsymbol{g}/N}\right| \\ &= \left|\sum_{\boldsymbol{k}\in\mathcal{L}} \widehat{f}(\boldsymbol{k})\right| \\ &\le \sum_{\boldsymbol{k}\in\mathcal{L}} |\widehat{f}(\boldsymbol{k})|.\end{aligned}$$

We can use the bound on the Fourier coefficients from the previous section to obtain

$$\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{\ell=0}^{N-1} f(\{\ell\boldsymbol{g}/N\})\right| \le C_{\alpha,s}\|f\|_\alpha \sum_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|^{-\alpha}.$$

The last sum tells us to choose $\boldsymbol{g}$ such that only those $\boldsymbol{k} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ should satisfy $\boldsymbol{k}\cdot\boldsymbol{g} \equiv 0 \pmod N$ for which $|\bar{k}_1\cdots\bar{k}_s|^{-\alpha}$ is small. Indeed, one can show that there are $\boldsymbol{g}$ such that $\sum_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|^{-\alpha} = \mathcal{O}(N^{-\alpha}(\log N)^{\alpha s})$.

One way to show the last claim is the following (we do not give the details here, just an outline, see [9, Chapter 5] for more information): Let

$$\rho = \min_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|. \tag{1}$$

We call ρ the figure of merit. Then the largest term in $\sum_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|^{-\alpha}$ is given by $\rho^{-\alpha}$. One can now show that the sum $\sum_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|^{-\alpha}$ is dominated by its largest term. Indeed, there are bounds

$$\rho^{-\alpha} \le \sum_{\boldsymbol{k}\in\mathcal{L}} |\bar{k}_1\cdots\bar{k}_s|^{-\alpha} \le C'_{\alpha,s}\rho^{-\alpha}(\log\rho)^{\alpha s}, \tag{2}$$

see [9, Chapter 5]. Further there is a result which states that there exists a $\boldsymbol{g} \in \{1,\ldots,N-1\}^s$ such that $\rho \ge c_s N$. Together with (2) this yields the result.

In the following we use a similar approach for numerical integration using digital nets. Instead of considering Fourier series, we now consider Walsh series and lattice rules are replaced by quasi-Monte Carlo rules based on digital nets. Before we can explain this theory we introduce the necessary concepts in the next section.

3 Preliminaries

In the following we introduce the digital construction scheme and Walsh functions. For simplicity we only consider the case where the base b is a prime.

3.1 The digital construction scheme

The construction of the point set used here is based on the concept of digital nets introduced by Niederreiter, see [9].

Definition 1. Let b be a prime and let $n, m, s \geq 1$ be integers. Let $C_1, \ldots, C_s$ be $n \times m$ matrices over the finite field $\mathbb{F}_b$ of order b. Now we construct b^m points in $[0,1)^s$: for $0 \leq h \leq b^m - 1$ let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the b-adic expansion of h. Identify h with the vector $\mathbf{h} = (h_0, \ldots, h_{m-1})^\top \in \mathbb{F}_b^m$, where $\top$ means the transpose of the vector (note that we write $\mathbf{h}$ for vectors in the finite field $\mathbb{F}_b^m$ and $\boldsymbol{h}$ for vectors of integers or real numbers). For $1 \leq j \leq s$ multiply the matrix C_j by $\mathbf{h}$, i.e.,

$$C_j \mathbf{h} =: (y_{j,1}(h), \ldots, y_{j,n}(h))^\top \in \mathbb{F}_b^n,$$

and set

$$x_{h,j} := \frac{y_{j,1}(h)}{b} + \cdots + \frac{y_{j,n}(h)}{b^n}.$$

The point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is called a digital net (over $\mathbb{F}_b$) (with generating matrices $C_1, \ldots, C_s$).

For $n, m = \infty$ we obtain a sequence $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$, which is called a digital sequence (over $\mathbb{F}_b$) (with generating matrices $C_1, \ldots, C_s$).

Niederreiter's concept of a digital (t, m, s)-net and a digital (t, s)-sequence will appear as a special case in the subsequent section. Further, the digital nets considered below all satisfy $n \geq m$.

For a digital net with generating matrices $C_1, \ldots, C_s$ let $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s)$ be the dual net given by

$$\mathcal{D} = \{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} : C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0}\},$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s)$ with $k_j = \kappa_{j,0} + \kappa_{j,1} b + \cdots$ and $\kappa_{j,i} \in \{0, \ldots, b-1\}$ we define $\mathbf{k}_j = (\kappa_{j,0}, \ldots, \kappa_{j,n-1})^\top$.

3.2 Walsh functions

Let the real number $x \in [0,1)$ have base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$, with $0 \leq x_i < b$ and where infinitely many x_i are different from $b-1$. For $k \in \mathbb{N}$, $k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_\nu b^{a_\nu - 1}$, $a_1 > \cdots > a_\nu > 0$ and $0 < \kappa_1, \ldots, \kappa_\nu < b$, we define the kth Walsh function by

$$\mathrm{wal}_k(x) = \omega_b^{\kappa_1 x_{a_1} + \cdots + \kappa_\nu x_{a_\nu}},$$

where $\omega_b = \mathrm{e}^{2\pi \mathrm{i}/b}$. For $k = 0$ we set $\mathrm{wal}_0(x) = 1$.

For a function $f : [0,1] \to \mathbb{R}$ we define the kth Walsh coefficient of f by

$$\widehat{f}(k) = \int_0^1 f(x)\overline{\mathrm{wal}_k(x)}\,\mathrm{d}x$$

and we can form the Walsh series

$$f(x) \sim \sum_{k=0}^{\infty} \widehat{f}(k)\mathrm{wal}_k(x).$$

Note that throughout the paper Walsh functions and digital nets are defined using the same prime number b.

4 Higher order convergence of smooth functions using generalized digital nets

In this section we present the ideas behind higher order quasi-Monte Carlo rules based on generalized digital nets.

4.1 Decay of the Walsh coefficients of smooth functions

We will focus mainly on $s = 1$ in this section, the case $s > 1$ is a natural extension as we consider tensor product spaces of functions. We do not give all the details, but provide an heuristic approach. The simplest exposition of the result presented in this subsection which contains all the details may be found in [6].

We now prove a bound on the Walsh coefficients of smooth functions. Note that we cannot differentiate the Walsh series of a function f, since the Walsh functions are piecewise constant and have therefore jumps. But we can use the second approach based on integration by parts, as was done for Fourier series above. Let $J_k(x) = \int_0^x \mathrm{wal}_k(t)\,\mathrm{d}t$, then

$$\begin{aligned}\widehat{f}_{\mathrm{wal}}(k) &= \int_0^1 f(x)\overline{\mathrm{wal}_k(x)}\,\mathrm{d}x \\ &= [f(x)J_k(x)]_{x=0}^1 - \int_0^1 f'(x)J_k(x)\,\mathrm{d}x \\ &= -\int_0^1 f'(x)J_k(x)\,\mathrm{d}x, \end{aligned} \tag{3}$$

as $\int_0^1 \overline{\mathrm{wal}_k(x)}\,\mathrm{d}x = 0$.

As for Fourier series, we would now like to relate the Walsh coefficient $\widehat{f}_{\mathrm{wal}}(k)$ to some Walsh coefficient of f'. For Fourier series this happened naturally, but here we obtain the function J_k. The way to proceed now is to obtain the Walsh series expansion of J_k, which will allow us to relate the kth Walsh coefficient of f to some Walsh coefficients of f'.

We need the following lemma which was first shown in [8] and appeared in many other papers (see for example [5] for a more general version). The following notation will be used throughout the paper: $k' = k - \kappa_1 b^{a_1-1}$, and hence $0 \le k' < b^{a_1-1}$.

Lemma 1. *For* $k \in \mathbb{N}$ *let* $J_k(x) = \int_0^x \overline{\mathrm{wal}_k(t)}\,\mathrm{d}t$. *Then*

$$J_k(x) = b^{-a_1}\Bigg((1-\omega_b^{-\kappa_1})^{-1}\overline{\mathrm{wal}_{k'}(x)} + (1/2 + (\omega_b^{-\kappa_1}-1)^{-1})\overline{\mathrm{wal}_k(x)} + \sum_{c=1}^{\infty}\sum_{\vartheta=1}^{b-1} b^{-c}(\omega_b^{\vartheta}-1)^{-1}\overline{\mathrm{wal}_{\vartheta b^{a_1+c-1}+k}(x)}\Bigg).$$

For $k=0$, *i.e.,* $J_0(x) = \int_0^x 1\,\mathrm{d}t = x$, *we have*

$$J_0(x) = 1/2 + \sum_{c=1}^{\infty}\sum_{\vartheta=1}^{b-1} b^{-c}(\omega_b^{\vartheta}-1)^{-1}\overline{\mathrm{wal}_{\vartheta b^{c-1}}(x)}. \tag{4}$$

We also need the following elementary lemma.

Lemma 2. *For any* $0 < \kappa < b$ *we have*

$$|1-\omega_b^{-\kappa}|^{-1} \le \frac{1}{2\sin\frac{\pi}{b}} \quad \textit{and} \quad |1/2 + (\omega_b^{-\kappa}-1)^{-1}| \le \frac{1}{2\sin\frac{\pi}{b}}.$$

Let $k \in \mathbb{N}$ with $k = \kappa_1 b^{a_1-1} + \cdots + \kappa_\nu b^{a_\nu-1}$, where $0 < \kappa_1,\ldots,\kappa_\nu < b$ and $a_1 > \cdots > a_\nu > 0$. Further let $k^{(1)} = \kappa_2 b^{a_2-1} + \cdots + \kappa_\nu b^{a_\nu-1}$, $k^{(2)} = \kappa_3 b^{a_3-1} + \cdots + \kappa_\nu b^{a_\nu-1}$, and $k^{(\tau)} = \kappa_{\tau+1} b^{a_{\tau+1}-1} + \cdots + \kappa_\nu b^{a_\nu-1}$ for $0 \le \tau < \nu$ and $k^{(\nu)} = 0$. It is also convenient to define the following function:

$$\mu_\alpha(k) = \begin{cases} a_1 + \cdots + a_{\min(\alpha,\nu)} & \text{for } k > 0, \\ 0 & \text{for } k = 0. \end{cases}$$

Substituting the Walsh series for J_k in (3) we obtain approximately

$$\begin{aligned}\widehat{f}_{\mathrm{wal}}(k) &\approx -b^{-a_1}(1-\omega_b^{-\kappa_1})^{-1}\int_0^1 f'(x)\overline{\mathrm{wal}_{k'}(x)}\,\mathrm{d}x \\ &= -b^{-a_1}(1-\omega_b^{-\kappa_1})^{-1}\widehat{f'}_{\mathrm{wal}}(k^{(1)}).\end{aligned}$$

In actuality we obtain an infinite sum on the right hand side, but the main term is the first one, the remaining terms can be dealt with, see [6] for the details.

We can repeat the last step τ times until either $f^{(\tau)}$ is not differentiable anymore, or $k^{(\tau)} = 0$, that is, we can repeat it $\min(\alpha,\nu)$ times. Hence

$$\begin{aligned}
\widehat{f}_{\mathrm{wal}}(k) &\approx b^{-a_1}(\omega_b^{-\kappa_1}-1)^{-1}\widehat{f'}_{\mathrm{wal}}(k^{(1)}) \\
&\approx b^{-a_1-a_2}\prod_{i=1}^{2}(\omega_b^{-\kappa_i}-1)^{-1}\widehat{f''}_{\mathrm{wal}}(k^{(2)}) \\
&\vdots \\
&\approx b^{-a_1-\cdots-a_{\min(\alpha,\nu)}}\prod_{i=1}^{\min(\alpha,\nu)}(\omega_b^{-\kappa_i}-1)^{-1}\widehat{f}^{(\min(\alpha,\nu))}_{\mathrm{wal}}(k^{(\min(\alpha,\nu))}).
\end{aligned}$$

Taking the absolute value and using some estimation we obtain

$$\begin{aligned}
|\widehat{f}_{\mathrm{wal}}(k)| &\lessapprox b^{-a_1-\cdots-a_{\min(\alpha,\nu)}}\prod_{i=1}^{\min(\alpha,\nu)}|\omega_b^{-\kappa_i}-1|^{-1}|\widehat{f}^{(\min(\alpha,\nu))}_{\mathrm{wal}}(k^{(\min(\alpha,\nu))})| \\
&\le \frac{b^{-\mu_\alpha(k)}}{(2\sin\pi/b)^{\min(\alpha,\nu)}}|\widehat{f}^{(\min(\alpha,\nu))}_{\mathrm{wal}}(k^{(\min(\alpha,\nu))})| \\
&\le \frac{b^{-\mu_\alpha(k)}}{(2\sin\pi/b)^{\min(\alpha,\nu)}}\int_0^1\left|f^{(\min(\alpha,\nu))}(x)\right|\,\mathrm{d}x,
\end{aligned}$$

where we used $|\widehat{f}^{(\min(\alpha,\nu))}_{\mathrm{wal}}(k^{(\min(\alpha,\nu))})| = \left|\int_0^1 f^{(\min(\alpha,\nu))}(x)\overline{\mathrm{wal}_{k^{(\min(\alpha,\nu))}}(x)}\,\mathrm{d}x\right| \le \int_0^1\left|f^{(\min(\alpha,\nu))}(x)\right| |\overline{\mathrm{wal}_{k^{(\min(\alpha,\nu))}}(x)}|\,\mathrm{d}x = \int_0^1\left|f^{(\min(\alpha,\nu))}(x)\right|\,\mathrm{d}x$.

Thus if f is α times differentiable, we obtain

$$|\widehat{f}_{\mathrm{wal}}(k)| \lessapprox C_f b^{-\mu_\alpha(k)}.$$

By some modification of the above approach, see [6], it can be shown that the constant C_f, which depends on f, can be replaced by a constant which depends only on α and b (but not on f) and the norm of f, i.e., we have

$$|\widehat{f}_{\mathrm{wal}}(k)| \lessapprox C_{\alpha,b}\|f\|_\alpha b^{-\mu_\alpha(k)}.$$

The same holds for dimensions $s > 1$, see [1, 5, 6], where the constant additionally depends on the dimension s

$$|\widehat{f}(\boldsymbol{k})| \lessapprox C_{\alpha,b,s}\|f\|_\alpha b^{-\mu_\alpha(\boldsymbol{k})},$$

where $\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(k_1)+\cdots+\mu_\alpha(k_s)$ for $\boldsymbol{k} = (k_1,\ldots,k_s)$. For some values of b, this constant $C_{\alpha,b,s}$ goes to 0 exponentially as s increases, see [1, 6].

Thus we have now achieved an analogous result to the decay of the Fourier coefficients of smooth functions and we can now begin to investigate numerical integration.

4.2 Numerical integration

This section is largely similar to Section 2.3. Again, we have the property

$$\frac{1}{b^m}\sum_{\ell=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_\ell) = \begin{cases} 1 & \text{if } C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0} \in \mathbb{F}_b^m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $k_j = k_{j,0} + k_{j,1} b + \cdots$, and $\mathbf{k}_j = (k_{j,0}, \ldots, k_{j,n-1})^\top$.

The set of all $\boldsymbol{k}$ for which the sum above is 1 is called the dual net $\mathcal{D}$, i.e.,

$$\mathcal{D} = \{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} : C_1^\top \mathbf{k}_1 + \cdots + C_s^\top \mathbf{k}_s = \mathbf{0} \in \mathbb{F}_b^m\}.$$

Using the Walsh series expansion of the function f we obtain

$$\begin{aligned} \left| \int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{\ell=0}^{b^m-1} f(\boldsymbol{x}_\ell) \right| &= \left| \widehat{f}_{\mathrm{wal}}(\mathbf{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{\ell=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_\ell) \right| \\ &= \left| \sum_{\boldsymbol{k}\in\mathcal{D}} \widehat{f}_{\mathrm{wal}}(\boldsymbol{k}) \right| \\ &\le \sum_{\boldsymbol{k}\in\mathcal{D}} |\widehat{f}_{\mathrm{wal}}(\boldsymbol{k})|. \end{aligned}$$

We can use the bound on the Walsh coefficients of the previous section to obtain

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{\ell=0}^{b^m-1} f(\boldsymbol{x}_\ell) \right| \le C_{\alpha,b,s} \|f\|_\alpha \sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}.$$

The last inequality separates the contribution of the function from the contribution of the quasi-Monte Carlo rule, i.e., $\|f\|_\alpha$ depends only on the function f but not on the quasi-Monte Carlo rule, whereas $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}$ depends only on the generating matrices of the digital net and not on the function itself (only on the smoothness of f; i.e., it is the same for all functions which have smoothness α). Therefore, when considering the integration error we can now focus on the term $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}$, which we do in the following subsection.

4.3 Generalized digital nets

The aim is now to find digital nets, i.e., generating matrices $C_1, \ldots, C_s \in \mathbb{F}_b^{n\times m}$ such that $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})} = \mathcal{O}(N^{-\alpha}(\log N)^{\alpha s})$, where the number of quadrature points $N = b^m$.

Roughly speaking, the sum $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}$ is dominated by its largest term. To find this largest term, define

$$\mu_\alpha^*(C_1, \ldots, C_s) = \min_{\boldsymbol{k}\in\mathcal{D}} \mu_\alpha(\boldsymbol{k}).$$

The dependence on the generating matrices $C_1, \ldots, C_s$ on the right hand side of the above equation is via the dual net $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s)$. The largest term in $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}$ is then $b^{-\mu_\alpha^*(C_1,\ldots,C_s)}$.

In order to achieve a convergence of almost $N^{-\alpha} = b^{-\alpha m}$ we must have that the largest term in $\sum_{\boldsymbol{k}\in\mathcal{D}} b^{-\mu_\alpha(\boldsymbol{k})}$ is also of this order, that is, we must have $\mu_\alpha^*(C_1,\dots,C_s) \approx \alpha m$ (or say $\mu_\alpha^*(C_1,\dots,C_s) > \alpha m - t$ for some constant t independent of m). That this condition is also sufficient is quite technical and was shown in [5, Lemma 5.2]. (The definition of $\mu_\alpha^*(C_1,\dots,C_s)$ is reminiscent of the figure of merit for lattice rules, see (1). For lattice rules an approach of proving the desired order of convergence was described in Subsection 2.3.)

So now which matrices $C_1,\dots,C_s \in \mathbb{F}_b^{n\times m}$ achieve $\mu_\alpha^*(C_1,\dots,C_s) \approx \alpha m$? Again we can use some analogy: The definition of $\mu_\alpha^*(C_1,\dots,C_s)$ is similar to the figure of merit ρ for lattice rules, or more precisely to $\log\rho$, which for classical digital nets is analogous to the strength of the digital net, that is, $m-t$. On the other hand, the classical case corresponds to $\alpha = 1$, hence one can expect a relationship between $\mu_1^*(C_1,\dots,C_s)$ and $m-t$.

Indeed, we have the following: Let $C_j = (\mathbf{c}_{j,1}^\top,\dots,\mathbf{c}_{j,n}^\top)^\top$, i.e., $\mathbf{c}_{j,\ell} \in \mathbb{F}_b^m$ is the ℓth row of C_j. Then the matrices $C_1,\dots,C_s$ generate a classical digital (t,m,s)-net if for all $i_1,\dots,i_s \ge 0$ with $i_1+\dots+i_s \le m-t$, the vectors

$$\mathbf{c}_{1,1},\dots,\mathbf{c}_{1,i_1},\dots,\mathbf{c}_{s,1},\dots,\mathbf{c}_{s,i_s}$$

are linearly independent over $\mathbb{F}_b$.

Now assume $C_1,\dots,C_s$ generate a classical digital (t,m,s)-net and that we are given a $\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}$ with $\mu_1(\boldsymbol{k}) \le m-t$. Let $i_j = \mu_1(k_j)$ for $j = 1,\dots,s$, then $C_1^\top \mathbf{k}_1 + \dots + C_s^\top \mathbf{k}_s$ is a linear combination of the vectors $\mathbf{c}_{1,1},\dots,\mathbf{c}_{1,i_1},\dots,\mathbf{c}_{s,1},\dots,\mathbf{c}_{s,i_s}$. As $\boldsymbol{k} \ne \mathbf{0}$ and $i_1+\dots+i_s \le m-t$, which implies that $\mathbf{c}_{1,1},\dots,\mathbf{c}_{1,i_1},\dots,\mathbf{c}_{s,1},\dots,\mathbf{c}_{s,i_s}$ are linearly independent, it follows that $C_1^\top\mathbf{k}_1+\dots+C_s^\top\mathbf{k}_s \ne \mathbf{0} \in \mathbb{F}_b^m$. Thus $\boldsymbol{k} \notin \mathcal{D}$. This shows that if $C_1,\dots,C_s$ generate a classical digital (t,m,s)-net and $\boldsymbol{k} \in \mathcal{D}$, then $\mu_1(\boldsymbol{k}) > m-t$. This is precisely the type of result described above which we also want to have for $\alpha > 1$.

So in the classical case $\alpha = 1$ we had some linear independence condition of the rows of the generating matrices which lead to the desired result. How can we generalize this linear independence condition to $\alpha > 1$? So we want to have that if $\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}$ with $\mu_\alpha(\boldsymbol{k}) \le \alpha m - t$, then the generating matrices should have linear independent rows such that $C_1^\top\mathbf{k}_1+\dots+C_s^\top\mathbf{k}_s \ne \mathbf{0} \in \mathbb{F}_b^m$. Let $\boldsymbol{k} = (k_1,\dots,k_s)$, where $k_j = \kappa_{j,1}b^{a_{j,1}-1}+\dots+\kappa_{j,\nu_j}b^{a_{j,\nu_j}-1}$, with $a_{j,1} > \dots > a_{j,\nu_j} > 0$ and $0 < \kappa_{j,1},\dots,\kappa_{j,\nu_j} < b$. First note that if $n < \alpha m - t$, then $\boldsymbol{k} = (b^n, 0,\dots,0) \in \mathcal{D}$, but $\mu_\alpha(\boldsymbol{k}) = n+1 \le \alpha m - t$. In order to avoid this problem we may choose $n = \alpha m$. Hence we may now assume that $a_{j,1} \le n = \alpha m$ for $j = 1,\dots,s$, as otherwise $\mu_\alpha(\boldsymbol{k}) > \alpha m$ already and no independence condition on the generating matrices is required in this case.

Now $C_1^\top\mathbf{k}_1+\dots+C_s^\top\mathbf{k}_s$ is a linear combination of the rows

$$\mathbf{c}_{1,a_{1,1}},\dots,\mathbf{c}_{1,a_{1,\nu_1}},\dots,\mathbf{c}_{s,a_{s,1}},\dots,\mathbf{c}_{s,a_{s,\nu_s}}.$$

Thus, if these rows are linearly independent, then $C_1^\top\mathbf{k}_1+\dots+C_s^\top\mathbf{k}_s \ne \mathbf{0} \in \mathbb{F}_b^m$, and therefore $\boldsymbol{k} \notin \mathcal{D}$.

Therefore, if $C_1, \ldots, C_s \in \mathbb{F}_b^{n \times m}$ are such that for all choices of $a_{j,1} > \cdots > a_{j,\nu_j} > 0$ for $j = 1, \ldots, s$, with

$$a_{1,1} + \cdots + a_{1,\min(\alpha,\nu_1)} + \cdots + a_{s,1} + \cdots + a_{s,\min(\alpha,\nu_s)} \le \alpha m - t,$$

the rows

$$\mathbf{c}_{1,a_{1,1}}, \ldots, \mathbf{c}_{1,a_{1,\nu_1}}, \ldots, \mathbf{c}_{s,a_{s,1}}, \ldots, \mathbf{c}_{s,a_{s,\nu_s}}$$

are linearly independent, then $\boldsymbol{k} \in \mathcal{D}$ implies that $\mu_\alpha(\boldsymbol{k}) > \alpha m - t$. (Note that we also include the case where some $\nu_j = 0$, in which case we just set $a_{j,1} + \cdots + a_{j,\min(\alpha,\nu_j)} = 0$.)

We can now formally define such digital nets for which the generating matrices satisfy such a property. The following definition is a special case of [5, Definition 4.3].

Definition 2. *Let $m, \alpha \ge 1$, and $0 \le t \le \alpha m$ be natural numbers. Let $\mathbb{F}_b$ be the finite field of prime order b and let $C_1, \ldots, C_s \in \mathbb{F}_b^{\alpha m \times m}$ with $C_j = (\mathbf{c}_{j,1}^\top, \ldots, \mathbf{c}_{j,\alpha m}^\top)^\top$. If for all $0 < a_{j,\nu_j} < \cdots < a_{j,1}$, where $0 \le \nu_j$ for all $j = 1, \ldots, s$, with*

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} a_{j,l} \le \alpha m - t$$

the vectors

$$\mathbf{c}_{1,a_{1,\nu_1}}, \ldots, \mathbf{c}_{1,a_{1,1}}, \ldots, \mathbf{c}_{s,a_{s,\nu_s}}, \ldots, \mathbf{c}_{s,a_{s,1}}$$

are linearly independent over $\mathbb{F}_b$, then the digital net with generating matrices $C_1, \ldots, C_s$ is called a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_b$.

The need for a more general definition in [5] arises as we assume therein that the smoothness α of the integrand is not known, so one cannot choose $n = \alpha m$ in this case.

We have seen so far that a digital $(t, \alpha, 1, \alpha m \times m, s)$-net used as quadrature points in a quasi-Monte Carlo rule will yield a convergence of the integration error of order $N^{-\alpha}(\log N)^{\alpha s}$ for integrands with $\|f\|_\alpha < \infty$.

The remaining question now is: do digital $(t, \alpha, 1, \alpha m \times m, s)$-nets for all given $\alpha, s \ge 1$ and some fixed t (which may depend on α and s but not on m) exist for all $m \in \mathbb{N}$? An affirmative answer to this question will be given in the next subsection.

4.4 Construction of generalized digital nets

In this subsection we present explicit constructions of digital $(t, \alpha, 1, \alpha m \times m, s)$-nets. The basic construction principle appeared first in [4] and was slightly modified in [5]. The construction requires a parameter d, which, in case the smoothness of the integrand α is known, should be chosen as $d = \alpha$. In this subsection we present this construction and a bound on the t-value, but we assume that α is known explicitly and hence choose $d = \alpha$.

Let $C_1, \ldots, C_{s\alpha}$ be the generating matrices of a digital $(t', m, s\alpha)$-net; we recall that many explicit examples of such generating matrices are known, see e.g., [7, 9, 10, 11, 12, 13, 17] and the references therein. As we will see later, the choice of the underlying $(t', m, s\alpha)$-net has a direct impact on the bound on the t-value of the digital $(t, \alpha, 1, \alpha m \times m, s)$-net. Let $C_j = (\mathbf{c}_{j,1}^\top, \ldots, \mathbf{c}_{j,m}^\top)^\top$ for $j = 1, \ldots, s\alpha$; i.e., $\mathbf{c}_{j,l}$ are the row vectors of C_j. Now let the matrix $C_j^{(\alpha)}$ be made of the first rows of the matrices $C_{(j-1)\alpha+1}, \ldots, C_{j\alpha}$, then the second rows of $C_{(j-1)\alpha+1}, \ldots, C_{j\alpha}$, and so on. The matrix $C_j^{(\alpha)}$ is then an $\alpha m \times m$ matrix; i.e., $C_j^{(\alpha)} = (\mathbf{c}_{j,1}^{(\alpha)}, \ldots, \mathbf{c}_{j,\alpha m}^{(\alpha)})^\top$, where $\mathbf{c}_{j,l}^{(\alpha)} = \mathbf{c}_{u,v}$ with $l = (v-j)\alpha + u$, $1 \le v \le m$, and $(j-1)\alpha < u \le j\alpha$ for $l = 1, \ldots, \alpha m$ and $j = 1, \ldots, s$.

To give the idea why this construction works we may consider the case $s = 1$. Let $\alpha > 1$. To simplify the notation we drop the j (which denotes the coordinate) from the notation for a moment. Let $C^{(\alpha)}$ be constructed from a classical digital (t', m, α)-net with generating matrices $C_1, \ldots, C_\alpha$ as described above. Let $\alpha m \ge a_1 > a_2 > \cdots > a_\nu \ge 1$. Then we need to consider the row vectors $\mathbf{c}_{a_1}^{(\alpha)}, \ldots, \mathbf{c}_{a_\nu}^{(\alpha)}$. Now by the construction above, the vector $\mathbf{c}_{a_1}^{(\alpha)}$ may stem from any of the generating matrices $C_1, \ldots, C_\alpha$. W.l.o.g. assume that $\mathbf{c}_{a_1}^{(\alpha)}$ stems from C_1, i.e., it is the i_1th row of C_1, where $i_1 = \lceil a_1/\alpha \rceil$. Next consider $\mathbf{c}_{a_2}^{(\alpha)}$. This row vector may again stem from any of the matrices $C_1, \ldots, C_\alpha$. If $\mathbf{c}_{a_2}^{(\alpha)}$ also stems from C_1, then $\lceil a_2/\alpha \rceil < i_1$. If not, we may w.l.o.g. assume that it stems from C_2. Indeed, it will be the i_2th row of C_2, where $i_2 = \lceil a_2/\alpha \rceil$. We continue in this fashion and define numbers $i_3, i_4, \ldots, i_l$, where $1 \le l \le \alpha$. Further we set $i_{l+1} = \cdots = i_\alpha = 0$. Then by the (t', m, α)-net property of $C_1, \ldots, C_\alpha$, it follows that $\mathbf{c}_{a_1}^{(\alpha)}, \ldots, \mathbf{c}_{a_\nu}^{(\alpha)}$ are linearly independent provided that $i_1 + \cdots + i_\alpha \le m - t'$. Hence, if we choose t such that $a_1 + \cdots + a_{\min(\alpha,\nu)} \le \alpha m - t$ implies that $i_1 + \cdots + i_\alpha \le m - t'$ for all admissible choices of $a_1, \ldots, a_\nu$, then the digital $(t, \alpha, 1, \alpha m \times m, 1)$-net property of $C^{(\alpha)}$ follows.

Note that $i_1 = \lceil a_1/\alpha \rceil$ and $i_l \le \lceil a_l/\alpha \rceil$ for $l = 2, \ldots, \alpha$. Thus

$$\begin{aligned} i_1 + \cdots + i_\alpha &\le \lceil a_1/\alpha \rceil + \cdots + \lceil a_\alpha/\alpha \rceil \\ &\le (a_1 + \cdots + a_\alpha + \alpha(\alpha - 1))/\alpha \\ &= \frac{a_1 + \cdots + a_\alpha}{\alpha} + \alpha - 1 \\ &\le m - t/\alpha + \alpha - 1. \end{aligned}$$

Thus, if we choose t such that $m - t/\alpha + \alpha - 1 \le m - t'$, then the result follows. Simple algebra then shows that

$$t = \alpha t' + \alpha(\alpha - 1)$$

will suffice.

A more general and improved result is given in the following which is a special case of [5, Theorem 4.11], with an improvement for some cases from [2] (a proof of this result can be found in [2, 4]).

Theorem 1. *Let $\alpha \geq 1$ be a natural number and let $C_1, \ldots, C_{s\alpha}$ be the generating matrices of a digital $(t', m, s\alpha)$-net over the finite field $\mathbb{F}_b$ of prime order b. Let $C_1^{(\alpha)}, \ldots, C_s^{(\alpha)}$ be defined as above. Then the matrices $C_1^{(\alpha)}, \ldots, C_s^{(\alpha)}$ are the generating matrices of a digital $(t, \alpha, 1, \alpha m \times m, s)$-net over $\mathbb{F}_b$ with*

$$t = \alpha \min\left(m, t' + \left\lfloor \frac{s(\alpha - 1)}{2} \right\rfloor\right).$$

This shows that digital $(t, \alpha, 1, \alpha m \times m, s)$-nets exist for all $\alpha, m, s \geq 1$ with t bounded independently of m. Indeed, also the dependence of t on α and s is known from [2]: namely $t \asymp \alpha^2 s$.

Geometrical properties of digital $(t, \alpha, 1, \alpha m \times m, s)$-nets and their generalization were shown in [2]. In the following section we show pictures of those properties.

5 Geometrical properties of generalized digital nets

In this section we describe geometrical properties of generalized digital nets. The generating matrices $C_1^{(2)}$ and $C_2^{(2)}$ for the digital net shown in Figure 1 are obtained from the classical digital $(1, 4, 4)$-net with the following generating matrices:

$$C_1 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, C_2 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}, C_3 = \begin{pmatrix} 1&1&1&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}, C_4 = \begin{pmatrix} 0&1&1&0 \\ 1&1&0&1 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}.$$

Using the construction principle from [4, 5] described above, we obtain

$$C_1^{(2)} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{pmatrix} \text{ and } C_2^{(2)} = \begin{pmatrix} 1&1&1&1 \\ 0&1&1&0 \\ 0&1&0&1 \\ 1&1&0&1 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}.$$

Theorem 1 implies that $C_1^{(2)}$, $C_2^{(2)}$ generate a digital $(4, 2, 1, 8 \times 4, 2)$-net. Upon inspection one can see that it is also a $(3, 2, 1, 8 \times 4, 2)$-net, but not a $(2, 2, 1, 8 \times 4, 2)$-net (the first two rows of $C_1^{(2)}$ and $C_2^{(2)}$ are linearly dependent).

Figure 2 shows that the point set is classical $(1, 4, 2)$-net. Indeed, this is true more generally: generalized digital nets are also classical digital nets (with the classical t-value usually a bit worse than the best classical nets known for the chosen parameters, which is understandable as generalized digital nets

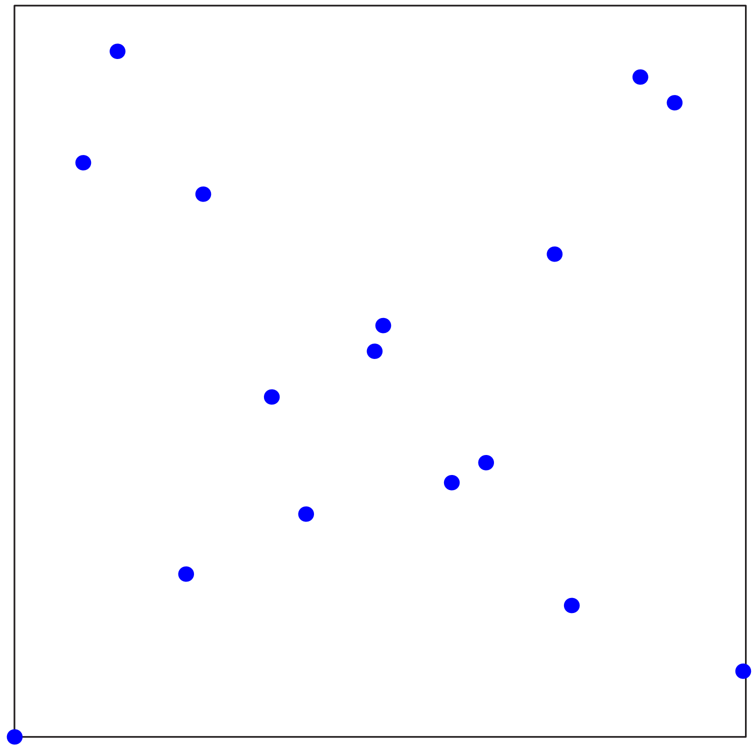


Fig. 1. A digital $(3, 2, 1, 8 \times 4, 2)$-net over $\mathbb{Z}_2$ which is also a classical digital $(1, 4, 2)$-net over $\mathbb{Z}_2$.

have some additional structure as we will see below) and are therefore also well distributed.

Figure 3 shows a partition of the square for which each union of the shaded rectangles contains exactly two points. Figures 4 and 5 show that also other partitions of the unit square are possible where each union of shaded rectangles contains the fair amount of points. Many other partitions of the square are possible where the point set always contains the fair amount of points in each union of rectangles, see [2], but there are too many of them to show them all here. Even in the simple case considered here there are 12 partitions possible, for each of which the point set is fair - this is quite remarkable since the point set itself has only 16 points (we exclude all those partitions for which the fairness would follow already from some other partition, otherwise there would be 34 of them). In the classical case we have 4 such partitions, all of which are shown in Figure 2. (The partitions from the classical case are included in the generalized case; so out of the 12 partitions 4 are shown

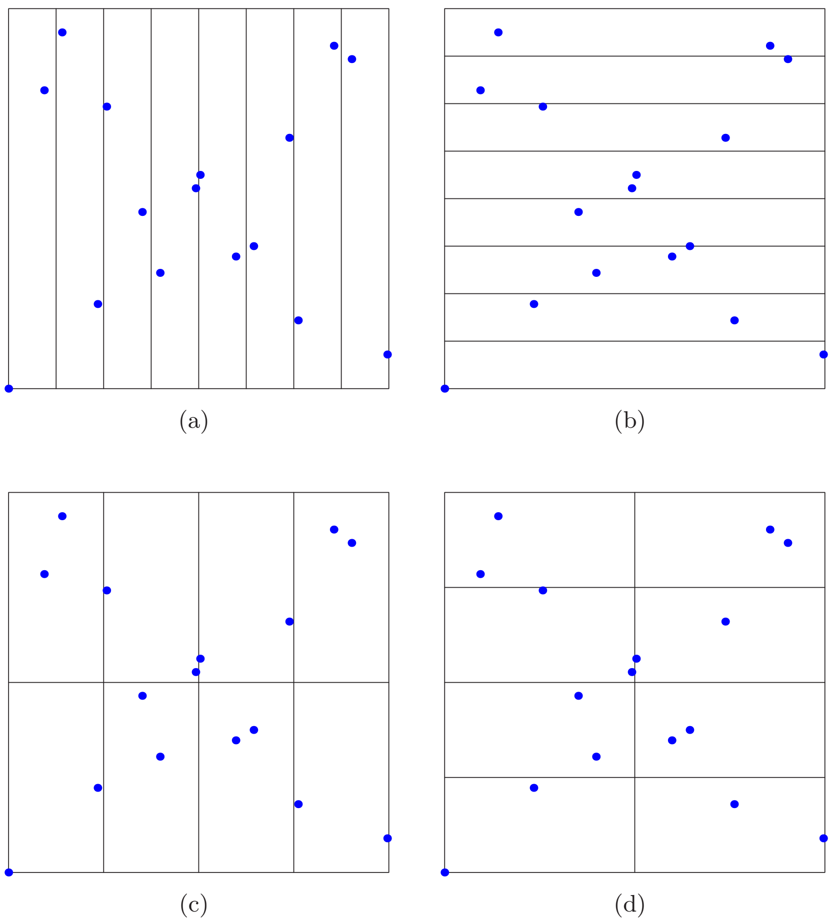
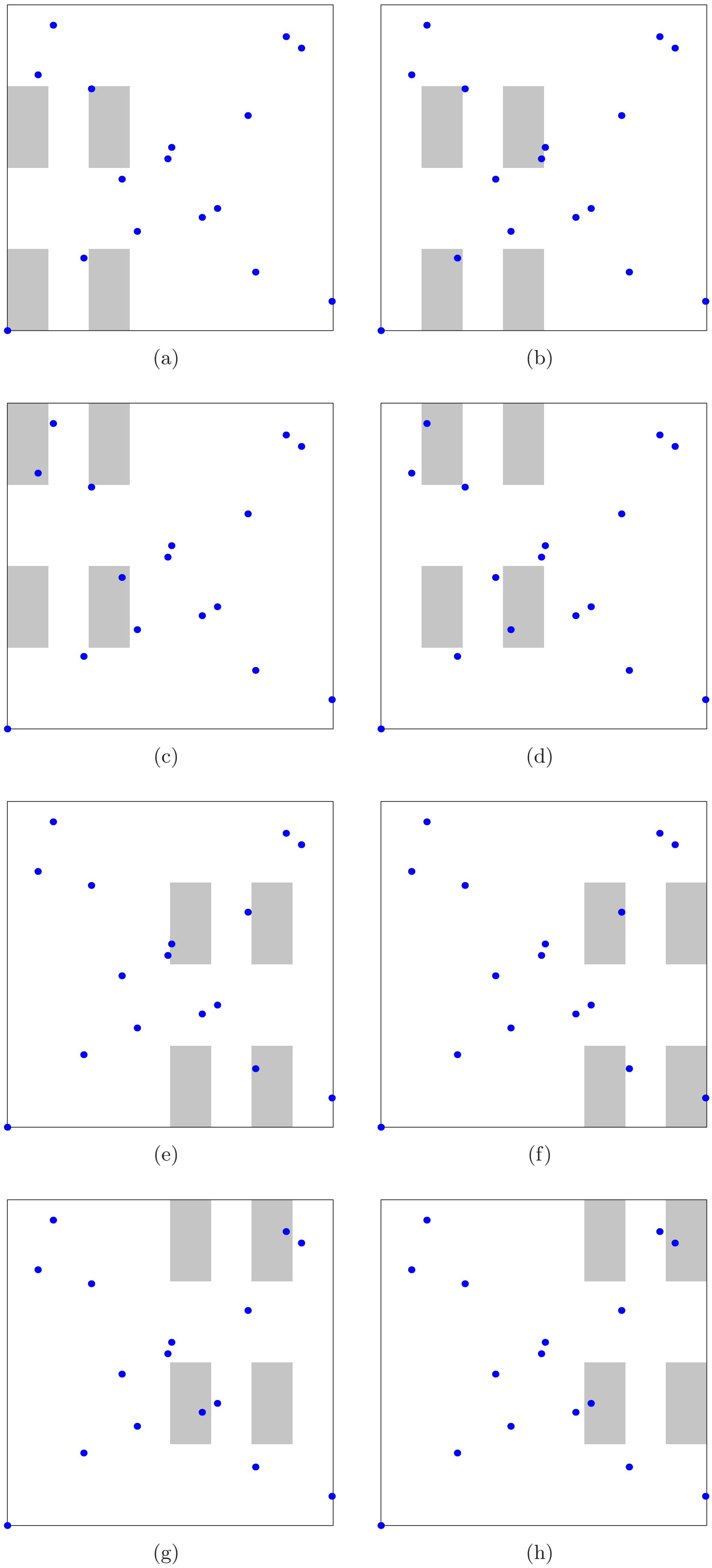


Fig. 2. The digital $(3, 2, 1, 8 \times 4, 2)$-net is also a digital $(1, 4, 2)$-net, as each partition of the unit square contains exactly two points.

in Figure 2, one is shown in Figure 3, one is shown in Figure 5 and one is indicated in Figure 4.)

The subsets of $[0, 1)^s$ which form a partition and which each have the fair amount of points are of the form:

$$J(\boldsymbol{a}_{\boldsymbol{\nu}}, \boldsymbol{d}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{d_{j,l}=0 \\ l \in \{1,\ldots,\alpha m\} \setminus \{a_{j,1},\ldots,a_{j,\nu_j}\}}}^{b-1} \left[\frac{d_{j,1}}{b} + \cdots + \frac{d_{j,n}}{b^{\alpha m}}, \frac{d_{j,1}}{b} + \cdots + \frac{d_{j,n}}{b^{\alpha m}} + \frac{1}{b^{\alpha m}}\right),$$



(a) (b)

(c) (d)

(e) (f)

(g) (h)

Fig. 3. The digital $(3, 2, 1, 8 \times 4, 2)$-net. The union of the shaded rectangles in each figure from (a) to (h) contains exactly two points.

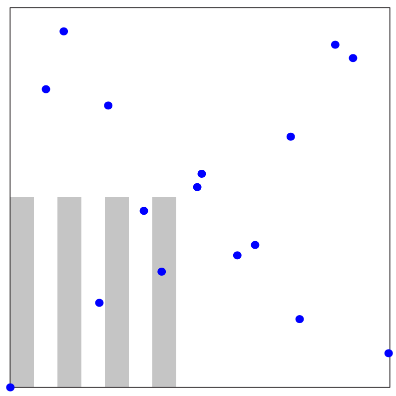


Fig. 4. Digital $(3, 2, 1, 8 \times 4, 2)$-net over $\mathbb{Z}_2$. The union of the shaded rectangles contains two points. As in Figure 3 one can also form a partition of the square with this type of rectangle where each union of rectangles contains two points.

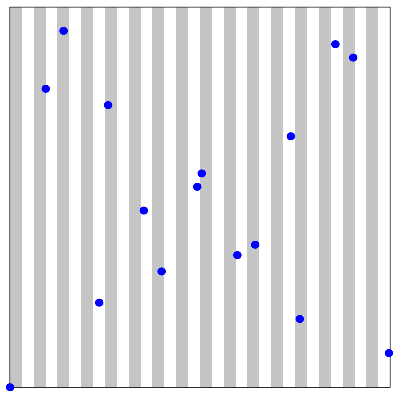


Fig. 5. Digital $(3, 2, 1, 8 \times 4, 2)$-net over $\mathbb{Z}_2$. The union of the shaded rectangles contains half the points.

where $b \geq 2$ is the base and where $\sum_{j=1}^{s} \sum_{l=1}^{\nu_j} a_{j,l} \leq \alpha m - t$. For $j = 1, \ldots, s$ we again assume $1 \leq a_{j,\nu_j} < \cdots < a_{j,1} \leq \alpha m$ in case $\nu_j > 0$ and $\{a_{j,1}, \ldots, a_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$. Further, we also use the following notation: $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s)$, $|\boldsymbol{\nu}|_1 = \sum_{j=1}^{s} \nu_j$, $\mathbf{a}_{\boldsymbol{\nu}} = (a_{1,1}, \ldots, a_{1,\nu_1}, \ldots, a_{s,1}, \ldots, a_{s,\nu_s})$, $\mathbf{d}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, and $\mathbf{d}_{\boldsymbol{\nu}} = (d_{1,i_{1,1}}, \ldots, d_{1,i_{1,\nu_1}}, \ldots, d_{s,i_{s,1}}, \ldots, d_{s,i_{s,\nu_s}})$, where the components $a_{j,l}$ and $d_{j,l}$, $l = 1, \ldots, \nu_j$, do not appear in the vectors $\mathbf{a}_{\boldsymbol{\nu}}$ and $\mathbf{d}_{\boldsymbol{\nu}}$ in case $\nu_j = 0$.

Figures 2, 3, 4, and 5 give only a few examples of unions of intervals for which each subset of the partition contains the right amount of points. As the $J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$, for fixed $\boldsymbol{\nu}$ and $\mathbf{a}_{\boldsymbol{\nu}}$ (with $\mathbf{d}_{\boldsymbol{\nu}}$ running through all possibilities) form a partition of $[0,1)^s$, it is clear that the right amount of points in $J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$ has to be $b^m \mathrm{Vol}(J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$. For example, the digital net in Figure 3 has 16 points and the partition consists of 8 different subsets $J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$, hence each $J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$ contains exactly $16/8 = 2$ points. (In general, the volume of $J(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{d}_{\boldsymbol{\nu}})$ is given by $b^{-|\boldsymbol{\nu}|_1}$, see [2].)

6 Geometrical numerical integration

The geometrical properties needed for numerical integration can be illustrated in the one dimensional case.

Assume $f : [0,1] \to \mathbb{R}$ is twice continuously differentiable. Then

$$f(x) = f(0) + \int_0^x f'(t) \,\mathrm{d}t = f(0) + x f'(0) + \int_0^1 (x-t)_+ f''(t) \,\mathrm{d}t, \qquad (5)$$

where $(x-t)_+$ is $x-t$ for $x \ge t$ and 0 otherwise.

Let $x_1, \ldots, x_N \in [0,1]$, then using (5) we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{h=1}^N f(x_h) - \int_0^1 f(x)\,\mathrm{d}x \\
&= f(0) + f'(0)\frac{1}{N}\sum_{h=1}^N x_h + \frac{1}{N}\sum_{h=1}^N \int_0^1 (x_h - t)_+ f''(t)\,\mathrm{d}t \\
&\quad - f(0) - f'(0)\int_0^1 x\,\mathrm{d}x - \int_0^1\int_0^1 (x-t)_+ f''(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= f'(0)\left[\frac{1}{N}\sum_{h=1}^N x_h - \int_0^1 x\,\mathrm{d}x\right] \\
&+ \int_0^1 f''(t)\left[\frac{1}{N}\sum_{h=1}^N (x_h - t)_+ - \int_0^1 (x-t)_+\,\mathrm{d}x\right]\mathrm{d}t.
\end{aligned}$$

Taking the absolute value of the integration error we obtain

$$\left|\frac{1}{N}\sum_{h=1}^N f(x_h) - \int_0^1 f(x)\,\mathrm{d}x\right| \le \left[|f'(0)| + \int_0^1 |f''(t)|\,\mathrm{d}t\right] \sup_{0\le t\le 1} |\Delta_N(t)|,$$

where

$$\Delta_N(t) = \left|\frac{1}{N}\sum_{h=1}^N (x_h - t)_+ - \int_0^1 (x-t)_+\,\mathrm{d}x\right|.$$

The factor $|f'(0)| + \int_0^1 |f''(t)|\,\mathrm{d}t$ is a seminorm of the function f and the factor $\sup_{0\le t\le 1} |\Delta_N(t)|$ measures properties of the quadrature points $x_1, \ldots, x_N$.

For example, for $t = 0$ the quadrature rule would numerically integrate the function x and
$$\Delta_N(0) = \frac{1}{N}\sum_{h=1}^{N} x_h - \int_0^1 x \, \mathrm{d}x$$
is the integration error.

In order to obtain a convergence of $N^{-2+\delta}$, $\delta > 0$, our quadrature points should be chosen such that $\Delta_N(t) = \mathcal{O}(N^{-2+\delta})$ for any $0 \le t \le 1$. Any equidistant quadrature points only yield $\sup_{0\le t\le 1} |\Delta_N(t)| = \mathcal{O}(N^{-1})$. The points from a digital $(t, \alpha, 1, \alpha m \times m, s)$-net are not equidistant for $\alpha > 1$, but introduce some cancelation effect as we explain in the following.

Consider Figure 6. We assume we want to numerically integrate the function x (then $\Delta(0)$ would be the integration error) using a $(t, 2, 1, 2m \times m, 1)$-net (where $b = 2$). This function is relevant as it appears in the upper bound (the case where $0 < t \le 1$ is similar.) Assume we want to put two points in the interval $[0, 1/2)$, such that one point is in $[0, 1/4)$ and another one is in $[1/4, 1/2)$, as illustrated in Figure 6(a). Then we get some integration error for the point x_1 in $[0, 1/4)$ of the form $e_1 = x_1 - \int_0^{1/4} \mathrm{d}x$ and another integration error for the point x_2 in $[1/4, 1/2)$ of the form $e_2 = x_2 - \int_{1/4}^{1/2} x \, \mathrm{d}x$. The integration error for the interval $[0, 1/2)$ is then the sum of the two errors $e_1 + e_2$. If both e_1 and e_2 have the same sign then the absolute value of error $|e_1 + e_2|$ for the integral $\int_0^{1/2} x \, \mathrm{d}x$ increases, whereas if they have opposite signs then we get some cancelation effect and the absolute value of the error, $|e_1 + e_2|$, decreases.

We can partition each of the intervals $[0, 1/4)$ and $[1/4, 1/2)$ again into two intervals to obtain $[0, 1/8)$ and $[1/8, 1/4)$ on the one hand and $[1/4, 3/8)$ and $[3/8, 1/2)$ on the other hand, see Figure 6(b). Next we put two points in the interval $[0, 1/2)$: In Figure 6(c) one point is in the interval $[0, 1/8)$ and the other one in $[3/8, 1/2)$ and in Figure 6(d) one is in $[1/8, 1/4)$ and one in $[1/4, 3/8)$. In both cases, when considering the integral $\int_0^{1/2} x \, \mathrm{d}x$, we get some cancelation effect: in Figure 6(c) the point in $[0, 1/8)$ underestimates the integral $\int_0^{1/4} x \, \mathrm{d}x$, whereas the point in $[3/8, 1/2)$ overestimates the integral $\int_{1/4}^{1/2} x \, \mathrm{d}x$. Similarly for Figure 6(d). On the other hand, in Figure 6(e) both points underestimate the corresponding integral and in Figure 6(f) both points overestimate the corresponding integral - hence the integration errors add up in this case.

So we started out saying that we want to have one point in the black interval in Figure 6(g) and one in the white. But to get some cancelation effect we also want to have that one point is in the black part in Figure 6(h) and one in the white part.

But such a structure is exhibited by the point set shown in Figure 1. Considering the projection of the point set onto the x-axis, Figure 2(c) shows that the same number of points is in the interval $[0, 1/4)$ as there is in $[1/4, 1/2)$.

Figures 3(a) and (b) on the other hand show that the same number of points is in $[0,1/8) \cup [1/4,3/8)$ as there is in $[1/8,1/4) \cup [3/8,1/2)$. Therefore this point set shows the desired cancelation effect which allows us to obtain a convergence beyond $\mathcal{O}(N^{-1+\delta})$.

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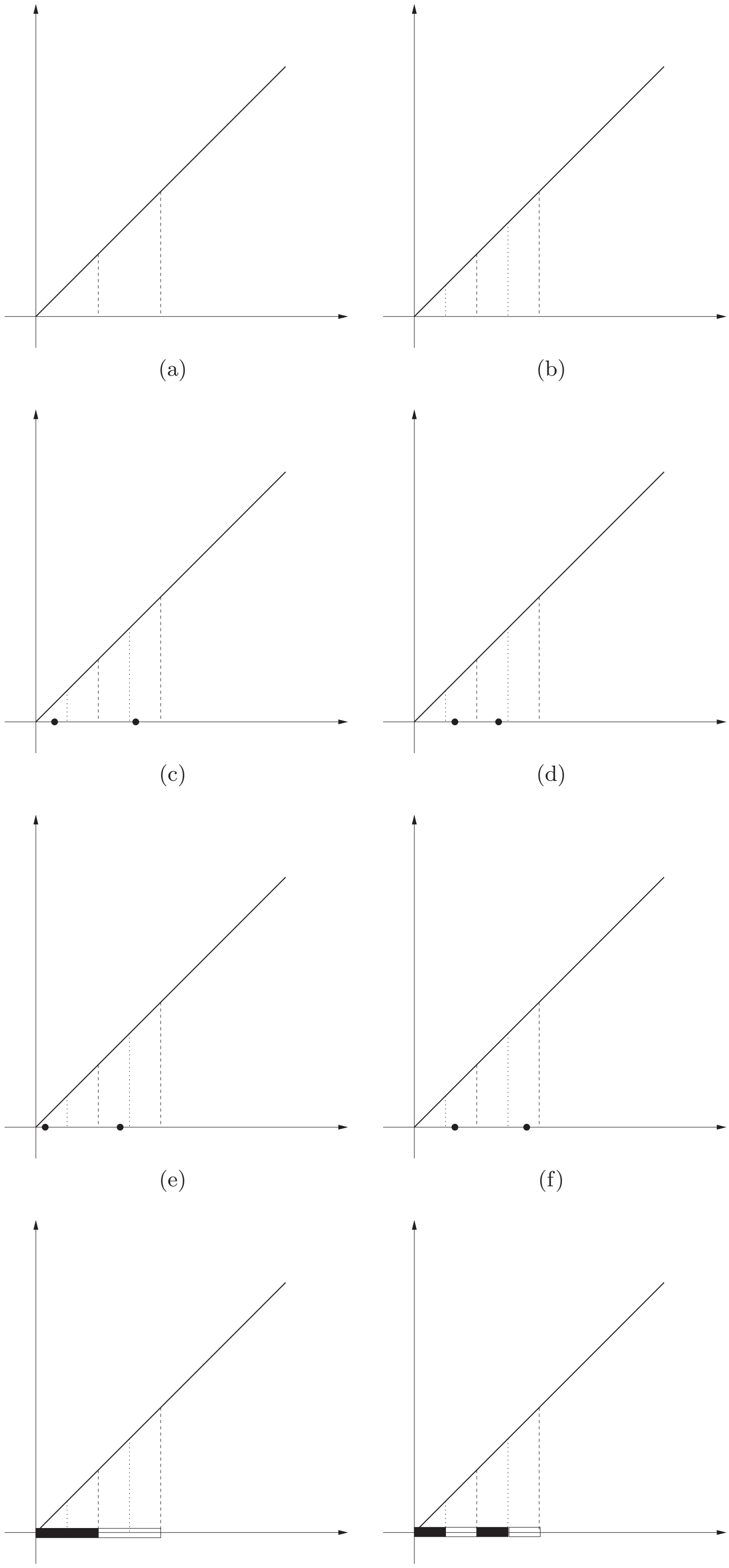

(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 6. Geometrical numerical integration.

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